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Prepared by: **K.J.Badcock**

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The Influence of Structural Parameter Variability on Aeroelastic LCO Amplitude

K.J. Badcock* and H.H. Khodaparast†

Department of Engineering,

University of Liverpool,

L69 3BX, United Kingdom.

I. Introduction

Previous work by the authors has considered the impact of uncertainty in structural parameters on aeroelastic stability. Particular consideration has been given to how large dimensional systems like those arising from computational fluid dynamics can be made tractable.[?]

There is considerable practical interest in evaluating the impact of structural parameter variability on limit cycle responses.[2] This is a demanding task since nonlinearity must be involved in some form (which tends to render model reduction through methods like proper orthogonal decomposition less effective than for linear problems). The best current approach seems to be that described in reference.[3] This paper contains a good review of the current state of the art.

The current report has the objective of reconsidering the (nonlinear) model reduction presented in references[4,5] for application to parametric variability studies. The approach is to calculate the critical eigenspace of the linearised system and use this as a basis for model reduction. The full order system is manipulated using a Taylor expansion which is then projected onto the critical eigenspace basis. The extra step here, after reimplementing the method, is to add the influence of the uncertain parameter to the Taylor Series. This allows the reduced model (two degrees-of-freedom) to be precomputed, and then exploited for the variability analysis. The formulation is described and then results are presented. The test case used has some limitations, but the results are sufficiently encouraging to motivate develop of the method for large scale problems.

II. Formulation

A. Full order Model

Assume incompressible and inviscid flow. Then Fung[6] expressed the coefficients of lift and pitching moment in terms of integrals of derivatives of the pitch and plunge response, derived from the Wagner function. Lee[7] showed that the integrals can be removed by defining the variables

$$w_1 = \int_0^t e^{-\epsilon_1(t-\sigma)}\alpha(\sigma)d\sigma \tag{1}$$

$$w_2 = \int_0^t e^{-\epsilon_2(t-\sigma)}\alpha(\sigma)d\sigma \tag{2}$$

$$w_3 = \int_0^t e^{-\epsilon_1(t-\sigma)}h(\sigma)d\sigma \tag{3}$$

*Professor, SMAIAA, corresponding author. Tel. : +44(0)151 794 4847 , Fax. : +44(0)151 794 6841, Email : K.J.Badcock@liverpool.ac.uk

†Research Associate

$$w_4 = \int_0^t e^{-\epsilon_2(t-\sigma)} h(\sigma) d\sigma \tag{4}$$

with $\epsilon_1 = 0.0455$ and $\epsilon_2 = 0.3$ being constants. Then the equations of motion are written as

$$\frac{d\mathbf{x}}{dt} = \mathbf{R} \tag{5}$$

where $\mathbf{x} = [\alpha, \dot{\alpha}, h, \dot{h}, w_1, w_2, w_3, w_4]^T$. Here α is the pitch angle about the elastic axis (positive nose up) and h is the vertical displacement of the elastic axis (positive down and non-dimensionalised by the semi-chord b). The residual vector is given by

$$\mathbf{R} = \begin{bmatrix} x_2 \\ g(c_0 H - d_0 P) \\ x_4 \\ g(-c_1 H + d_1 P) \\ x_1 - \epsilon_1 x_5 \\ x_1 - \epsilon_2 x_6 \\ x_3 - \epsilon_1 x_7 \\ x_3 - \epsilon_2 x_8 \end{bmatrix} \tag{6}$$

where $g = 1/(d_0 c_1 - c_0 d_1)$ and

$$P = c_2 x_4 + c_3 x_2 + c_4 x_3 + c_5 x_3^3 + c_{51} x_3^5 + c_6 x_1 + c_7 x_5 + c_8 x_6 + c_9 x_7 + c_{10} x_8$$

$$H = d_2 x_4 + d_3 x_2 + d_4 x_3 + d_5 x_3^3 + d_{51} x_3^5 + d_6 x_1 + d_7 x_5 + d_8 x_6 + d_9 x_7 + d_{10} x_8$$

with the effect of the initial conditions on P and H neglected since we are interested in the long time dynamics. Here the coefficients are defined in terms of the structural parameters as

$$\begin{aligned}
c0 &= 1 + 1/\mu \\
c1 &= x_\alpha - a_h/\mu \\
c2 &= 2(1 - \psi_1 - \psi_2)/\mu \\
c3 &= (1 + (1 - 2a_h)(1 - \psi_1 - \psi_2))/\mu \\
c4 &= \omega^2/\bar{u}^2 + 2(\psi_1\epsilon_1 + \psi_2\epsilon_2)/\mu \\
c5 &= \beta_{h3}\omega^2/\bar{u}^2 \\
c51 &= \beta_{h5}\omega^2/\bar{u}^2 \\
c6 &= 2((1 - \psi_1 - \psi_2) + (0.5 - a_h)(\psi_1\epsilon_1 + \psi_2\epsilon_2))/\mu \\
c7 &= 2\psi_1\epsilon_1(1 - (0.5 - a_h)\epsilon_1)/\mu \\
c8 &= 2\psi_2\epsilon_2(1 - (0.5 - a_h)\epsilon_2)/\mu \\
c9 &= -2\psi_1\epsilon_1\epsilon_1/\mu \\
c10 &= -2\psi_2\epsilon_2\epsilon_2/\mu
\end{aligned}$$

$$\begin{aligned}
d0 &= x_\alpha/(r_\alpha r_\alpha) - a_h/\mu_r^2 \\
d1 &= 1 + (1 + 8a_h a_h)/(8\mu_r^2) \\
d2 &= (1 - 2a_h)/(2\mu_r^2) - (1 + 2a_h)(1 - 2a_h)(1 - \psi_1 - \psi_2)/(2\mu_r^2) \\
d3 &= 1/\bar{u}^2 - (1 + 2a_h)(1 - \psi_1 - \psi_2)/\mu_r^2 - (1 + 2a_h)(1 - 2a_h)(\psi_1\epsilon_1 + \psi_2\epsilon_2)/(2\mu_r^2) \\
d4 &= \beta_{\alpha 1}/\bar{u}^2 \\
d41 &= \beta_{\alpha 5}/\bar{u}^2 \\
d5 &= -(1 + 2a_h)(1 - \psi_1 - \psi_2)/\mu_r^2 \\
d6 &= -(1 + 2a_h)(\psi_1\epsilon_1 + \psi_2\epsilon_2)/\mu_r^2 \\
d7 &= -(1 + 2a_h)\psi_1\epsilon_1(1 - (0.5 - a_h)\epsilon_1)/\mu_r^2 \\
d8 &= -(1 + 2a_h)\psi_2\epsilon_2(1 - (0.5 - a_h)\epsilon_2)/\mu_r^2 \\
d9 &= (1 + 2a_h)\psi_1\epsilon_1\epsilon_1/\mu_r^2 \\
d10 &= (1 + 2a_h)\psi_2\epsilon_2\epsilon_2/\mu_r^2
\end{aligned}$$

Here $\mu_r^2 = \mu r_\alpha^2$, r_α is the radius of gyration, $\omega = \omega_h/\omega_\alpha$ is the ratio of the uncoupled plunging (ω_h) to pitching (ω_α) frequencies, $\bar{u} = u/b\omega_\alpha$ is the reduced velocity, μ is the aerofoil to air mass ratio, the elastic axis is at a location $a_h b$ from the mid chord and the mass centre is a distance $x_\alpha b$ from the elastic axis. The coefficients β_{h3}, β_{h5}, $\beta_{\alpha 3}$ and $\beta_{\alpha 5}$ are the elastic restoring force constants, relative to the linear restoring term, for the plunging and pitching springs, with the subscript 3 denoting the cubic term and the subscript 5 the quintic term. The constants $\psi_1 = 0.165$ and $\psi_2 = 0.335$ arise from the Wagner function used to model the aerodynamics.

The matlab function

```
ode45
```

is used to solve the ordinary differential equation (5). The initial condition used is $\mathbf{x} = [1^o, 0, 0, 0, 0, 0, 0, 0]^T$.

B. Jacobian Matrices

The model reduction discussed below is based on a Taylor expansion of the residual up to third order terms. This requires the calculation of matrix vector products against the first, second and third Jacobian matrices.

The first Jacobian is given by

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{21} & L_{2,3} & A_{23} & L_{5,2} & L_{7,7} & L_{8,8} & L_{9,9} & L_{10,10} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ A_{41} & M_{2,3} & A_{43} & M_{5,2} & M_{7,7} & M_{8,8} & M_{9,9} & M_{10,10} \\ 1 & 0 & 0 & 0 & -\epsilon_1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -\epsilon_2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -\epsilon_1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -\epsilon_2 \end{bmatrix} \tag{7}$$

where $L_{i,j} = s_1 d_i + s_2 c_j$ and $M_{i,j} = s_3 d_i + s_4 c_j$, with $s_1 = gc_0$, $s_2 = -gd_0$, $s_3 = -gc_1$ and $s_4 = gd_1$, and

$$A_{21} = L_{3,6} + s_1(3d_4 x_1^2 + 5d_{41} x_1^4)$$
$$A_{23} = L_{6,4} + s_2(3c_5 x_3^2 + 5c_{51} x_3^4)$$
$$A_{41} = L_{3,6} + s_3(3d_4 x_1^2 + 5d_{41} x_1^4)$$
$$A_{43} = L_{6,4} + s_4(3c_5 x_3^2 + 5c_{51} x_3^4)$$

The second Jacobian is given by $B = [B_{ijk}]$ where

$$B_{211} = 6s_1 d_4 x_1 + 20 s_1 d_{41} x_1^3$$
$$B_{233} = 6s_2 c_5 x_3 + 20 s_2 c_{51} x_3^3$$
$$B_{411} = 6s_3 d_4 x_1 + 20 s_3 d_{41} x_1^3$$
$$B_{433} = 6s_4 c_5 x_3 + 20 s_4 c_{51} x_3^3$$

and the remaining terms are zero.

The third Jacobian is given by $C = [C_{ijkl}]$ where

$$C_{2111} = 6s_1 d_4 + 60 s_1 d_{41} x_1^2$$
$$C_{2333} = 6s_2 c_5 + 60 s_2 c_{51} x_3^2$$
$$C_{4111} = 6s_3 d_4 + 60 s_3 d_{41} x_1^2$$
$$C_{4333} = 6s_4 c_5 + 60 s_4 c_{51} x_3^2$$

and the remaining terms are zero.

The derivatives of $\mathbf{R}$ with respect to the bifurcation parameter $\bar{u}$ are also required. The terms that depend on $\bar{u}$ are c_1, c_5, c_{51}, d_3, d_4 and d_{41}. The first derivatives are

$$\frac{\partial c_4}{\partial \bar{u}} = -\frac{2\omega^2}{\bar{u}^3}$$

$$\frac{\partial c_5}{\partial \bar{u}} = -\frac{2\beta_{h3}\omega^2}{\bar{u}^3}$$

$$\frac{\partial c_{51}}{\partial \bar{u}} = -\frac{2\beta_{h5}\omega^2}{\bar{u}^3}$$

$$\frac{\partial d_3}{\partial \bar{u}} = -\frac{2}{\bar{u}^3}$$

$$\frac{\partial d_4}{\partial \bar{u}} = -\frac{2\beta_{\alpha 3}}{\bar{u}^3}$$

$$\frac{\partial c_{51}}{\partial \bar{u}} = -\frac{2\beta_{\alpha 5}}{\bar{u}^3}$$

The Jacobian terms are obtained by substituting the expression for the derivative $\partial c_k/\partial \bar{u}$ for c_k in the residual or Jacobian terms given above.

Finally, the reduced model is also parameterised in the structural parameter (ω) which is assumed uncertain in this study. This requires the derivatives of the Residual with respect to this parameter to be calculated. This dependence is felt through the values of c_4, c_5 and c_{51}. The derivatives are given by

$$\frac{\partial c_4}{\partial \omega} = 2\omega/\bar{u}^2$$

$$\frac{\partial c_5}{\partial \omega} = 2\beta_{h3}\omega/\bar{u}^2$$

$$\frac{\partial c_{51}}{\partial \omega} = 2\beta_{h5}\omega/\bar{u}^2$$

The Jacobian terms are obtained by substituting the expression for the derivative $\partial c_k/\partial \omega$ for c_k in the residual or Jacobian terms given above.

The Taylor series expansion of the Residual function about the equilibrium $\mathbf{x}_0$ and the bifurcation value $\bar{u}_c$ is

$$\begin{aligned}\mathbf{R}(\mathbf{w},\bar{u}) \approx\; & \mathbf{R}(\mathbf{w}_0,\bar{u}_c) + A\mathbf{w}' + \frac{1}{2}B(\mathbf{w}',\mathbf{w}') + \frac{1}{6}C(\mathbf{w}',\mathbf{w}',\mathbf{w}') + \\ & \frac{\partial \mathbf{R}}{\partial \bar{u}}\bar{u}' + \frac{\partial A}{\partial \bar{u}}\mathbf{w}'\bar{u}' + \frac{1}{2}\frac{\partial B(\mathbf{w}',\mathbf{w}')}{\partial \bar{u}}\bar{u}' + \frac{1}{6}\frac{\partial C(\mathbf{w}',\mathbf{w}',\mathbf{w}')}{\partial \bar{u}}\bar{u}' + ... \\ & \frac{\partial \mathbf{R}}{\partial \omega}\omega' + \frac{\partial A}{\partial \omega}\mathbf{w}'\omega' + \frac{1}{2}\frac{\partial B(\mathbf{w}',\mathbf{w}')}{\partial \omega}\omega' + \frac{1}{6}\frac{\partial C(\mathbf{w}',\mathbf{w}',\mathbf{w}')}{\partial \omega}\omega' + ...\end{aligned} \tag{8}$$

where

$$B(\mathbf{x},\mathbf{y}) = \sum_{i,j} \frac{\partial^2 \mathbf{R}}{\partial x_i \partial y_j} x_i y_j$$

$$C(\mathbf{x},\mathbf{y},\mathbf{z}) = \sum_{i,j,k} \frac{\partial^3 \mathbf{R}}{\partial x_i \partial y_j \partial z_k} x_i y_j z_k,$$

$\mathbf{w}' = \mathbf{w} - \mathbf{w}_0$, $\bar{u}' = \bar{u} - \bar{u}_c$ and $\omega' = \omega - \omega_0$.

Note that a matrix free product can be used to avoid the need to form the Jacobian matrices explicitly as

$$A\mathbf{x}' \approx \frac{\mathbf{R}_1 - \mathbf{R}_{-1}}{2h} \tag{9}$$

$$B(\mathbf{x}',\mathbf{x}') \approx \frac{\mathbf{R}_1 - 2\mathbf{R}_0 + \mathbf{R}_{-1}}{h^2} \tag{10}$$

$$C(\mathbf{x}',\mathbf{x}',\mathbf{x}') \approx \frac{-\mathbf{R}_3 + 8\mathbf{R}_2 - 13\mathbf{R}_1 + 13\mathbf{R}_{-1} - 8\mathbf{R}_{-2} + \mathbf{R}_{-3}}{8h^3} \tag{11}$$

where $\mathbf{R}_i = \mathbf{R}(\mathbf{x}_0 + ih\mathbf{x}', \bar{u})$.

To allow the manipulation of terms like $B(x,y)$ and $C(x,y,z)$, the following identities are useful:

$$B(v,u) = \frac{1}{4}[B(v+u,v+u) - B(v-u,v-u)] \tag{12}$$

$$C(v,v,u) = \frac{1}{6}[C(v+u,v+u,v+u) - C(v-u,v-u,v-u) - 2C(u,u,u)] \tag{13}$$

$$B(u+v,u+v) = B(u,u) + 2B(u,v) + B(v,v) \tag{14}$$

$$C(u+v,u+v,u+v) = C(u,u,u) + 3C(u,u,v) + 3C(u,v,v) + C(v,v,v). \tag{15}$$

C. Linear Stability

The bifurcation value $\bar{u}_c$ is calculated by considering eigenvalues of the Jacobian matrix A. Lower $\bar{u}_{min}$ and upper $\bar{u}_{max}$ values are defined that bracket the critical value. This means that the largest real part eigenvalue of A changes sign within the interval of $[\bar{u}_{min}, \bar{u}_{max}$. The matlab function

```
eig
```

is used to calculate the largest real part eigenvalue. The bisection method is used to converge the interval onto the critical value. Note that this calculation provides solutions to the following eigenvalue problems:

$$A\mathbf{p} = i\omega\mathbf{p}$$

$$A\bar{\mathbf{p}} = -i\omega\bar{\mathbf{p}}$$
$$A^T\mathbf{q} = -i\omega\mathbf{q}$$
$$A^T\bar{\mathbf{q}} = i\omega\bar{\mathbf{q}}.$$

The eigenvectors can be scaled to give $<\mathbf{p},\mathbf{p}>= 1$, $<\mathbf{q},\mathbf{q}>= 1$ and $<\mathbf{q},\mathbf{p}>= 1$. It also follows that $<\mathbf{q},\bar{\mathbf{p}}>= 0$ since $<\mathbf{q},\bar{\mathbf{p}}>=<\mathbf{q},\frac{1}{-i\omega}A\bar{\mathbf{p}}>=\frac{1}{-i\omega}<A^T\mathbf{q},\bar{\mathbf{p}}>=\frac{-\omega}{-i\omega}<\mathbf{q},\bar{\mathbf{p}}>$. Note that the inner product is defined as $<\mathbf{p},\mathbf{q}>=\mathbf{q}^T\mathbf{p}$.

D. Model Reduction

The approach adopted is to project the system, expanded in the Taylor series, onto a basis formed by the critical eigenvectors. The full order unknowns are written as $\mathbf{w}' = z\mathbf{p} + \bar{z}\bar{\mathbf{p}} + \mathbf{y}$. Note that $<\mathbf{q},\mathbf{y}>= 0$ and $z =<\mathbf{q},\mathbf{w}'>$. The contribution $z\mathbf{p} + \bar{z}\bar{\mathbf{p}}$ represents the component of the full order solution which is in the (critical) space spanned by the critical eigenvector. Note that this term is real, as required. Also, the component $\mathbf{y}$ is the component which is not in the critical space. It is assumed in the following that the limit cycle response is dominated by the contribution in the critical space. The task is therefore to find the equation that describes the dynamics of the complex coordinate z. The centre manifold theorem states that the dynamics of the full system is dominated by the critical space in the vicinity of the bifurcation point, and this allows the size of $\mathbf{y}$ to be limited in this vicinity, leading to

$$\mathbf{y} = \frac{1}{2}k_{20}z^2 + k_{11}z\bar{z} + \frac{1}{2}k_{02}\bar{z}^2.$$

The values of the vector valued coefficients k_{ij} need to be computed and this is discussed below.

The inner product of Eq. (9) is taken with respect to $\bar{\mathbf{q}}$. The substitution $\mathbf{w}' = z\mathbf{p} + \bar{z}\bar{\mathbf{p}}$ is made and then terms are simplified. The time derivative term becomes

$$\frac{d\bar{\mathbf{q}}^T\mathbf{w}'}{dt} = \frac{dz}{dt}.$$

Since the expansion is done about an equilibrium, $\mathbf{R}(\mathbf{w}_0, \bar{u}_c) = 0$. The first order term in the expansion becomes

$$\bar{\mathbf{q}}^T A\mathbf{w}' = z\bar{\mathbf{q}}^T A\mathbf{p} + \bar{z}\bar{\mathbf{q}}^T A\bar{\mathbf{p}} + \bar{\mathbf{q}}A\mathbf{y} = i\omega z.$$

The second order term is simplified using Eq (14) to give

$$\bar{\mathbf{q}}^T B(\mathbf{w}', \mathbf{w}') = z^2 \bar{\mathbf{q}}^T B(\mathbf{p}, \mathbf{p}) + 2z\bar{z}\bar{\mathbf{q}}^T B(\mathbf{p}, \bar{\mathbf{p}}) + \bar{z}^2 \bar{\mathbf{q}}^T B(\bar{\mathbf{p}}, \bar{\mathbf{p}}) + z\bar{\mathbf{q}}^T B(\mathbf{p}, \mathbf{y}) + \bar{z}\bar{\mathbf{q}}^T B(\bar{\mathbf{p}}, \mathbf{y})$$

and the terms in $\mathbf{y}$ are substituted using the centre manifold reduction. Note that for the current problem the second Jacobian evaluated at a zero equilibrium is identically zero, and so these contributions are also zero in the reduced model. Finally, the third order term is simplified using Eq. (15) to give

$$\bar{\mathbf{q}}^T C(\mathbf{w}', \mathbf{w}', \mathbf{w}') = z^3 \bar{\mathbf{q}}^T C(\mathbf{p}, \mathbf{p}, \mathbf{p}) + 3z^2\bar{z}\bar{\mathbf{q}}^T C(\mathbf{p}, \mathbf{p}, \bar{\mathbf{p}}) + 3z\bar{z}^2\bar{\mathbf{q}}^T C(\mathbf{p}, \bar{\mathbf{p}}, \bar{\mathbf{p}}) + \bar{z}^3\bar{\mathbf{q}}^T C(\bar{\mathbf{p}}, \bar{\mathbf{p}}, \bar{\mathbf{p}})$$

with all terms involving $\mathbf{y}$ neglected since they are of at least fourth order in z. Finally, the parameters terms are simply projected onto $\mathbf{q}$.

The centre manifold reduction gives

$$z\bar{\mathbf{q}}^T B(\mathbf{p}, \mathbf{y}) = \frac{1}{2} z^3 \bar{\mathbf{q}}^T B(\mathbf{p}, k_{20}) + z^2\bar{z}\bar{\mathbf{q}}^T B(\mathbf{p}, k_{11}) + \frac{1}{2} z\bar{z}^2 \bar{\mathbf{q}}^T B(\mathbf{p}, k_{02})$$

and

$$\bar{z}\bar{\mathbf{q}}^T B(\bar{\mathbf{p}}, \mathbf{y}) = \frac{1}{2} z^2\bar{z}\bar{\mathbf{q}}^T B(\bar{\mathbf{p}}, k_{20}) + z\bar{z}^2\bar{\mathbf{q}}^T B(\bar{\mathbf{p}}, k_{11}) + \frac{1}{2}\bar{z}^3\bar{\mathbf{q}}^T B(\bar{\mathbf{p}}, k_{02}).$$

Equations for the coefficients k_{ij} are obtained in the following way. First, an equation for $\mathbf{y}$ is obtained by differentiating the definition $\mathbf{y} = \mathbf{w} - z\mathbf{p} - \bar{z}\bar{\mathbf{p}}$ with respect to time. The term arising from $\dot{\mathbf{w}}$ is obtained from the Taylor expansion of the residual of the original system. The second and third terms are obtained from the original equation projected onto $\mathbf{q}$ as simplified above. The resulting expression is simplified and second order terms in z retained. A periodic variation with frequency ω is assumed for $\mathbf{y}$, and this is used to simplify the time derivative. The centre manifold relations are then used to replace $\mathbf{y}$ with quadratic terms in z. Terms in z^2, $z\bar{z}$ and $\bar{z}^2$ are equated to yield equations for the coefficients k_{ij} as

$$(2i\omega I - A)k_{20} = H_{20}$$
$$-Ak_{11} = H_{11}$$
$$(-2i\omega I - A)k_{02} = H_{02}$$

where

$$H_{20} = B(\mathbf{p}, \mathbf{p}) - \mathbf{q}^T B(\mathbf{p}, \mathbf{p})\mathbf{p} - \mathbf{q}^T B(\mathbf{p}, \mathbf{p})\bar{\mathbf{p}}$$
$$H_{11} = B(\mathbf{p}, \bar{\mathbf{p}}) - \mathbf{q}^T B(\mathbf{p}, \bar{\mathbf{p}})\mathbf{p} - \mathbf{q}^T B(\mathbf{p}, \bar{\mathbf{p}})\bar{\mathbf{p}}$$
$$H_{02} = B(\bar{\mathbf{p}}, \bar{\mathbf{p}}) - \mathbf{q}^T B(\bar{\mathbf{p}}, \bar{\mathbf{p}})\mathbf{p} - \mathbf{q}^T B(\bar{\mathbf{p}}, \bar{\mathbf{p}})\bar{\mathbf{p}}.$$

Then collecting all of the terms, an equation for z is given by

$$\begin{aligned} \frac{dz}{dt} \quad = \quad & i\omega z + \frac{1}{2}G_{pp}z^2 + G_{p\bar{p}}z\bar{z} + \frac{1}{2}G_{\bar{p}\bar{p}}\bar{z}^2 + G_{30}z^3 + G_{21}z^2\bar{z} + G_{12}z\bar{z}^2 + G_{03}\bar{z}^3 \\ & + P_0 + P_p z + P_{\bar{p}}\bar{z} + (P_{pp}\bar{z}^2 + 2P_{p\bar{p}}z\bar{z} + P_{\bar{p}\bar{p}}\bar{z}^2)/2 + \\ & (P_{ppp}\bar{z}^3 + 3P_{pp\bar{p}}\bar{z}^2 z + 3P_{p\bar{p}\bar{p}}z\bar{z}^2 + P_{\bar{p}\bar{p}\bar{p}}\bar{z}^3)/6 \end{aligned} \qquad (16)$$

where

$$G_{30} = G_{ppp}/6 + G_{pk_{20}}/4,$$
$$G_{21} = G_{pp\bar{p}}/2 + G_{pk_{11}}/2 + G_{\bar{p}k_{20}}/4,$$
$$G_{12} = G_{p\bar{p}\bar{p}}/2 + G_{pk_{02}}/4 + G_{\bar{p}k_{11}}/2$$
$$G_{03} = G_{\bar{p}\bar{p}\bar{p}}/6 + G_{\bar{p}k_{02}}/4.$$

The notation means that $G_{xy} = B(x, y)$ and $G_{xyz} = C(x, y, z)$. Also, the terms arising from the expansion in the parameters are given by

$$P_0 = \frac{\partial \mathbf{R}}{\partial \bar{u}}\bar{u}' + \frac{\partial \mathbf{R}}{\partial \omega}\omega'$$
$$P_x = A_{\bar{u}}x\bar{u}' + A_\omega x\omega'$$
$$P_{xy} = B_{\bar{u}}(x, y)\bar{u}' + B_\omega(x, y)\omega'$$

and

$$P_{xyz} = C_{\bar{u}}(x, y, z)\bar{u}' + C_\omega(x, y, z)\omega'.$$

III. Results

A. Overview

The objectives of the results section are to

- evaluate the performance of the model reduction
- use the reduced model to calculate the spread of limit cycle amplitudes due to an uncertain structural parameter.

Note that the test case used is only a first test of the approach since (a) the second Jacobian terms are always zero and (b) the full order model only has dimension eight, making the model reduction computationally unnecessary.

The parameters are:

$$\mu = 100$$
$$x_\alpha = 0.25$$
$$a_h = -0.5$$
$$\omega = 0.2$$
$$\beta_{h3} = 0$$
$$\beta_{\alpha 3} = 3$$
$$\beta_{h5} = 0$$
$$\beta_{\alpha 5} = 0$$
$$r_\alpha = 0.5$$

This case features a supercritical bifurcation.

We consider the uncertain structural parameter here to be ω (which is the ratio of the natural frequencies of plunging to pitching. This parameter is assumed to have its probability described by a Gaussian distribution. We are interested in calculating the LCO variation with $\bar{u}$ for each realisation of ω. There are two ways of doing this. The first calculates the bifurcation point for the current value of ω, recomputes the reduced model and then uses it to calculate the growth in LCO amplitude for different values of $\bar{u}$. The second calculates the reduced model at the mean value of ω, and parameterises this model in ω. This allows the reduced model to be calculated once and then used for each realisation of ω. These scenarios are considered in turn.

B. Recalculation of the Reduced Model

The traces of the eigenvalues as $\bar{u}$ is changed are shown in figure 1. One complex conjugate pair crosses the imaginary axis, bringing about the bifurcation. The variation of the real and imaginary parts of the critical eigenvalue with $\bar{u}$ is shown in figure 2. The crossing happens for $\bar{u} \approx 6.25$. It is interesting to note that the imaginary part changes by around 5% in the post bifurcation range of interest for the variability study.

The value of ω is next considered to be uncertain around the deterministic value (0.2), with a Gaussian distribution assumed and a standard deviation of 0.02. The resulting normalised probability distribution for omega, generated by the matlab function

```
omegadist=normrnd(0.2,0.02,250,1);
```

and is shown in Fig. 3. The resulting normalised probability distribution for the bifurcation point is also shown in this figure.

Next the limit cycle predictions of the full and reduced models are compared for the deterministic value of $\bar{u}$. The limit cycle amplitudes are shown in figure 4. Excellent agreement is obtain, with only minor differences obtained even at high amplitudes (when the physical modelling would no longer be applicable in any case). Tests were made to determine the origin of the small error which appears. First, the full order model residual was replaced by the Taylor series expansion to third order terms (but without the projection onto the critical mode). A second test projected the full order solution onto the critical mode, and then reconstructed the full order solution back from this projection. In the first case the error introduced by the Taylor series is tested, and in the second case the error from the basis used after the bifurcartion point.

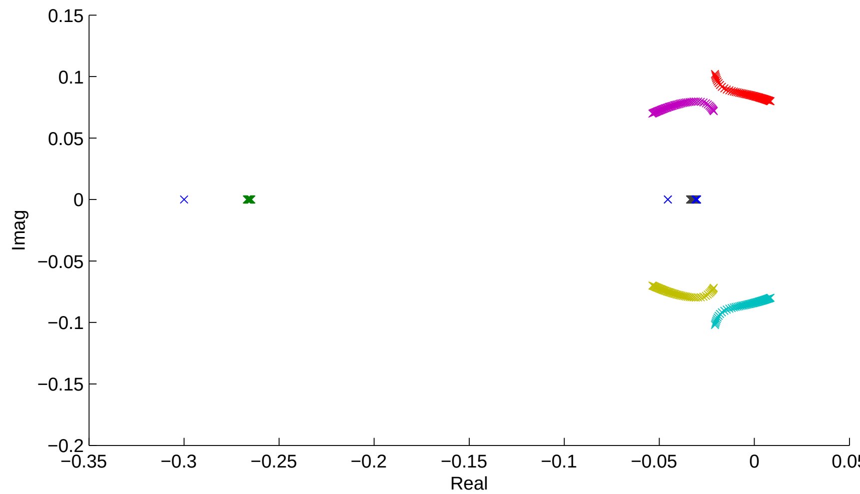


Figure 1. Root Locus

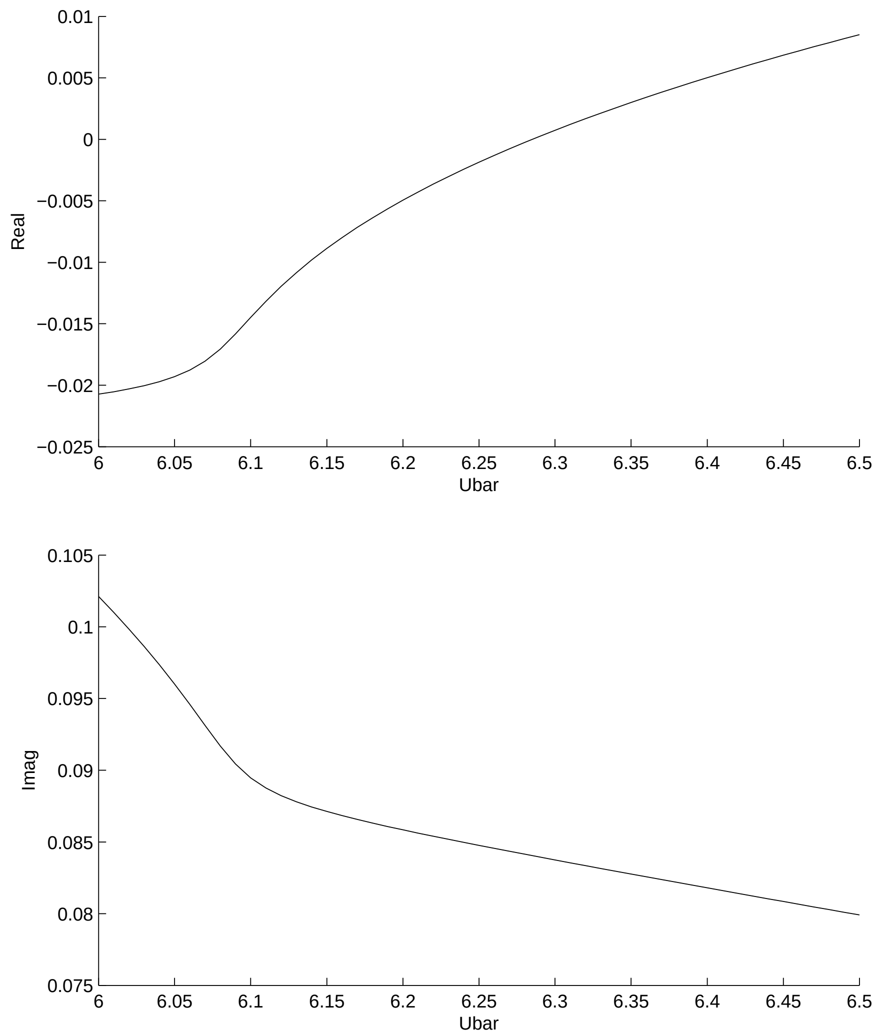
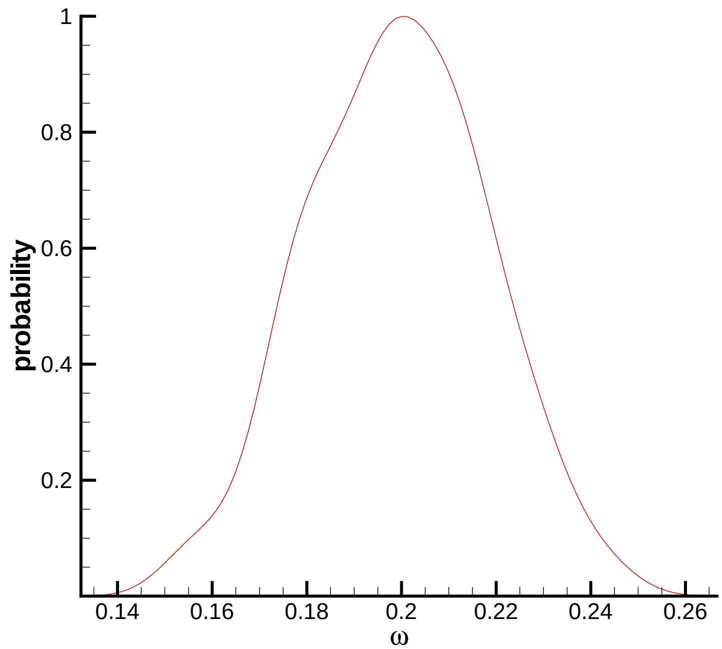


Figure 2. Critical Eigenvalue Traces



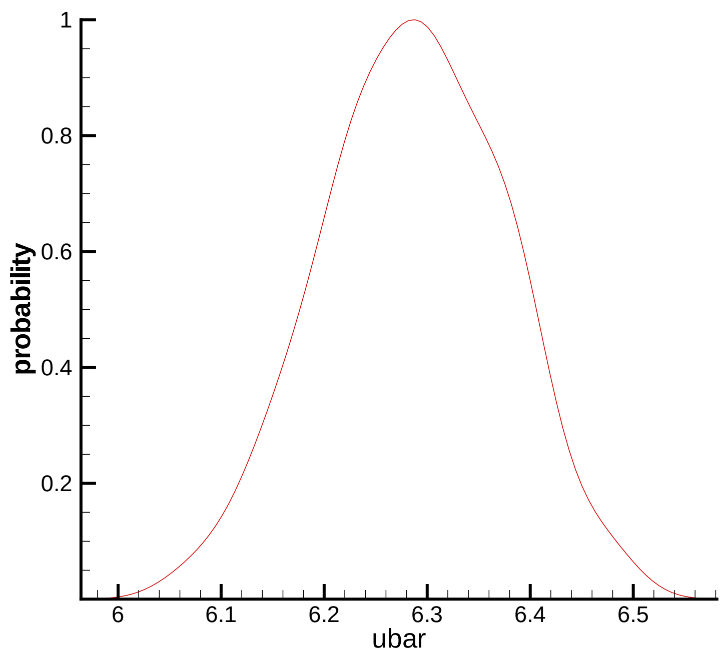


Figure 3. Probability Density Function for ω and Onset Reduced Velocity

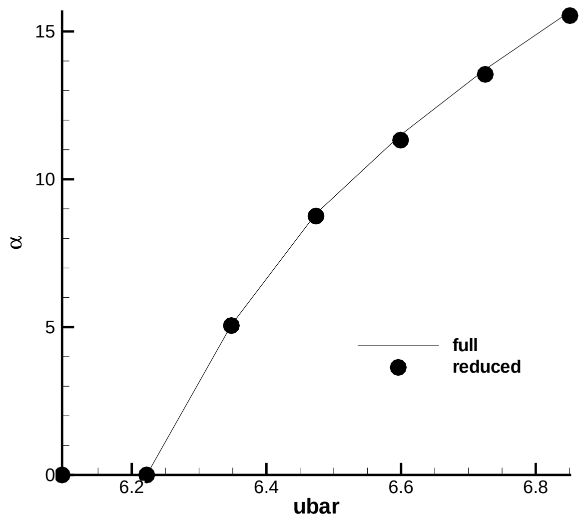


Figure 4. Limit Cycle Amplitude Growth

In both cases no error was observed. It was concluded that the series expansion in the parameter was the source of the small error.

Finally variability of the limit cycle growth due to the distribution in the value of ω was considered. The normalised probability distributions for several values of $\bar{u}$ are shown in Fig 5. Note that the reduced model is recalculated for each realisation of ω, with the saving in computational cost being that the same reduced model can be used for each value of $\bar{u}$. It can be seen that there is good agreement between the full order and reduced model predictions for the distributions. The spread of LCO amplitudes grows with the mean amplitude. At the highest mean amplitude shown the distribution develops a second peak. This seems to correspond to multiple solutions that are possible for the reduced model for some values of ω. This feature does not appear in the full order model and needs further investigation.

C. Parameterised Equation

The reduced model is now calculated at the mean value of ω, and the effect of the change in ω included through the Taylor series terms in this parameter. The comparison with the variation in LCO amplitude between this parameterised reduced model and the full order model. A Gaussian distribution with the same parameters as above was assumed for ω but this time 1000 samples was used. The probability distributions this time are not normal (at least for lower values of $\bar{u}$), and are shown in Fig. 6. For $\bar{u} = 6.4$ there is a peak at zero, indicating that the variation in ω can cause the bifurcation point to move through this value of $\bar{u}$. The parameterised reduced model does an excellent job of reproducing the full order probability distribution.

IV. Conclusions

The method presented has been used to evaluate the influence of structural parameter uncertainty on the LCO amplitude of an aerofoil free to move in pitch and plunge. The approach that developed a model based on the mean parameter bifurcation details, and then includes the influence of the reduced velocity and the uncertain structural parameters through the Taylor Series expansion is found to work very well for this case. Importantly, it has the potential to be generalised to very large dimension systems.

The test case used has a structural nonlinearity but is linear aerodynamically. Two important questions for the extension of the approach relate to how including more realistic modelling will impact on the method. First, the form of structural nonlinearity in aircraft structures needs to be considered to develop the next case with which to exercise the approach. Secondly, the type of nonlinearity (quadratic, cubic or higher)

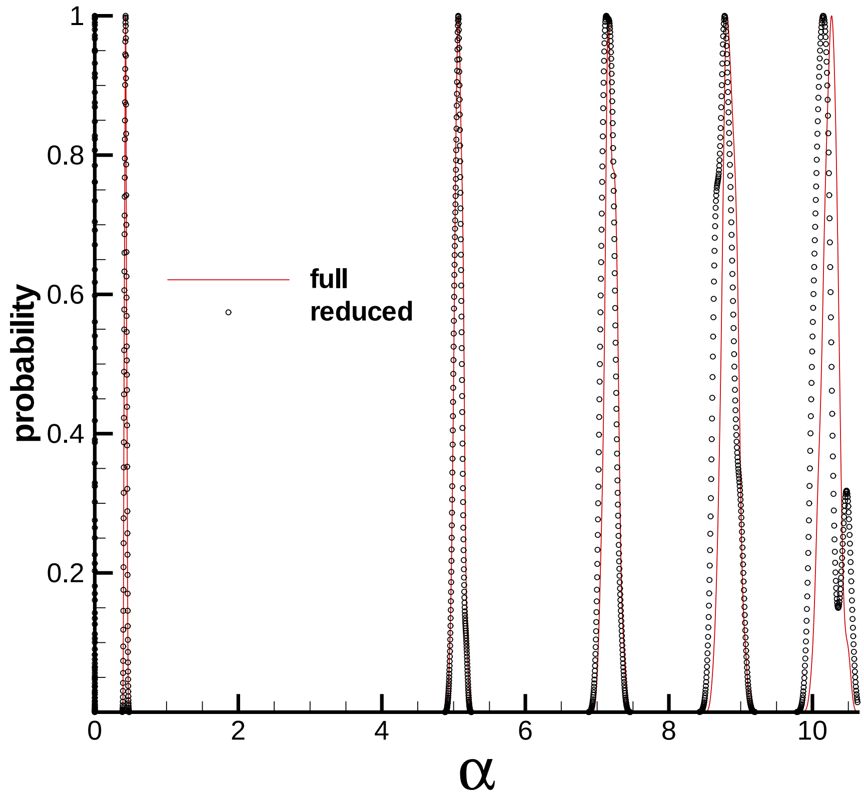
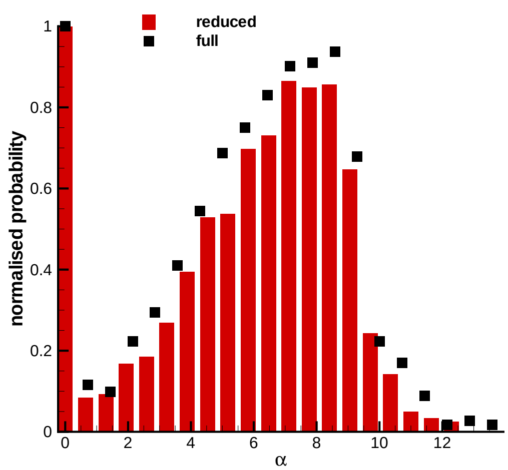
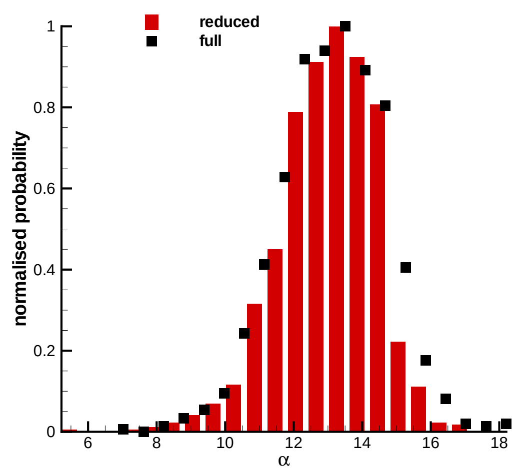


Figure 5. Limit Cycle Amplitude Growth Uncertainty



$\bar{u} = 6.4$



$\bar{u} = 6.7$

Figure 6. Normalised Probability Density Function for LCO pitch amplitude using the parametrised reduced order model.

that is associated with moving shock waves needs to be investigated. Note the results of references[4,5] that successfully used the reduced model approach to predict shock driven LCO and vortex driven wing rock. What is needed is a systematic evaluation of the Taylor Series convergence in these cases.

The variation of the critical eigenspace with the bifurcation parameter and reduced velocity is also very important. In the test case considered here the variation is very small, which contributes to the excellent results achieved. The variation in a realistic full scale case needs to be evaluated. It is also noted that the quadratic terms in the expansion in the case considered are zero, making this case unrepresentative in this respect.

A methodological extension would be to use normal form theory to simplify the reduced model. This may have the advantage of removing the need to calculate some terms in the model reduction.

Finally, a limitation that this test case has exposed is the lack of feasibility of the method to calculate sub-critical bifurcations. This is due to the need to represent fifth order effects in the Taylor series. This is feasible in the current case, but the need to form fifth Jacobian-vector products for large order systems makes the generalisation unattractive. New ideas are needed to cope with this case.

The current study was carried out using matlab scripts. Current work is tidying these up and rewriting them in the language python. It is intended that these python scripts will be made available on a web page.

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References

[1] Marques, S., Badcock, K.J., Khodaparast, H.H. and Mottershead, J.E., "CFD Based Aeroelastic Stability Predictions Under the Influence of Structural Variability", AIAA-2009-2324, May 2009.

[2] Bunton, R.W. and Denegri, C.M., "Limit Cycle Characteristics of Fighter Aircraft", Journal of Aircraft, Vol. 37, No. 5, 2000, 916-918.

[3] Beran, P.S., Pettit, C.L. and Millman, D.R., "Uncertainty Quatification of Limit-Cycle Oscillations", Journal of Computational Physics, Vol 217, 2006, 217-247.

[4] Woodgate, M.A. and Badcock, K.J., "Fast Prediction of Transonic Aeroelastic Stability and Limit Cycles", *AIAA Journal*, Vol. 45, No. 6, 2007, 1370-1381.

[5] Badcock, K.J., Woodgate, M.A., Allan, M.R. and Beran, P.S., "Wing-Rock Limit Cycle Oscillation Prediction Based on Computational Fluid Dynamics", *Journal of Aircraft*, Vol. 45, No. 3, 2008, pp 954-961.

[6] Fung, Y. C., "An Introduction to the heory of Aeroelasticity", Dover Publications, New York, 1969.

[7] Lee, B. H. K., Gong, L. and Wong, Y. S., "Analysis and computation of nonlinear dynamic response of a two-degree-of-freedom system and its application in aeroelasticity", *Journal of Fluids and Structures* Vol. 11, 1997, pp 225-246.