Efficient Methods in Stochastic Model Updating

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Abstract

Two efficient methods in stochastic model updating are developed and presented in this paper. The first method, a perturbation approach, is shown to be viable even when the correlation between system parameters and measurements is omitted. The requirement to calculate second order sensitivities then becomes unnecessary and this leads to considerable reduction in computational effort in practical engineering applications. The second method is based upon the minimisation of an objective function in two parts: 1- the Euclidian norm of the difference between mean values of measured data and analytical outputs vectors, and 2- the Frobenius norm of the difference between the covariance matrices of measured data and analytical outputs. The two methods are verified numerically and experimentally using multiple sets of plates with randomized thicknesses and masses.

1 Introduction

In model updating problems, system parameters are adjusted by minimizing the difference between measurements and predictions from a mathematical model [1,2]. The deterministic model updating problem changes to a stochastic model updating problem when the variability in measured vibration data is taken into account. It is important to include not only the variability in measurement signals due to noise, but also the variability that exists between nominally identical test structures, built in the same way from the same materials but with manufacturing and material variability [3,4,5] or disassembly and reassembly of the same structure [6]. Similar variability results from environmental erosion and damage [7,8,9].

Stochastic model updating problems require large amounts of computing time and we are therefore interested to determine what efficiencies can be gained by making various assumptions and simplifications. In this paper two iterative techniques are presented. The first method, a perturbation approach, is formulated in two ways, the first being a simplification of the second. In the simplified version, the correlation between the updated parameters and measured data is omitted. The need to compute the second order sensitivities is then found to be unnecessary. No significant deterioration is found in the estimated parameter distributions by using this assumption, which leads to a very considerable reduction in computational effort of great practical value in engineering. The second method, much simpler in concept, is based upon the minimisation of an objective function. The proposed objective function is the weighted sum of the Euclidian norm of the difference between mean values of measured data and analytical outputs vectors, and the Frobenius norm of the difference between the covariance matrices of the measured data and analytical outputs. Different optimization procedures may be used in order to minimise the objective function. The second method does not involve any assumption of statistical independence between the parameters and measurements.

The two methods are applied to numerical and experimental examples using multiple sets of plates with randomized thicknesses and masses. It is shown in numerical simulations that the both methods produce results that are equally acceptable to those produced by the methods which are computationally expensive. Issues of sample size are considered in a numerical example. The thickness of the plates is parameterised in four regions and the mean values and standard deviations are then identified by the first method in which the correlation between system parameters and measured data is omitted. The parameter distributions are shown to be significantly improved. In another study, the second method (employing a genetic algorithm) is used to estimate the distribution of uncertain masses on a flat plate. The identified and measured parameter distribution is found to be in good agreement.

2 The perturbation approach

The deterministic finite element model updating problem can be expressed as,

$$\boldsymbol{\theta}_{j+1} = \boldsymbol{\theta}_j + \mathbf{T}_j(\mathbf{z}_m - \mathbf{z}_j) \tag{1}$$

where $\mathbf{z}_j \in \Re^{n\times 1}$ is the vector of estimated output parameters (e.g. eigenvalues and eigenvectors), $\mathbf{z}_m \in \Re^{n\times 1}$ is the vector of measured data, $\boldsymbol{\theta} \in \Re^{m\times 1}$ is the vector of system parameters and $\mathbf{T}_j$ is a transformation matrix. In order to include the variability in measurements, the modal parameters are represented as,

$$\mathbf{z}_m = \overline{\mathbf{z}}_m + \Delta\mathbf{z}_m \tag{2}$$

$$\mathbf{z}_j = \overline{\mathbf{z}}_j + \Delta\mathbf{z}_j \tag{3}$$

where the overbar denotes mean values and $\Delta\mathbf{z}_m, \Delta\mathbf{z}_j \in \Re^{n\times 1}$ are vectors of random variables. The hyperellipses represented by $\{\overline{\mathbf{z}}_m, \mathrm{Cov}(\mathbf{z}_m, \mathbf{z}_m)\}$ and $\{\overline{\mathbf{z}}_j, \mathrm{Cov}(\mathbf{z}_j, \mathbf{z}_j)\}$ define the space of measurements and predictions respectively. Correspondingly the variability in physical parameters at the jth iteration is defined as,

$$\boldsymbol{\theta}_j = \overline{\boldsymbol{\theta}}_j + \Delta\boldsymbol{\theta}_j \tag{4}$$

and the stochastic model updating problem may then be cast as,

$$\overline{\boldsymbol{\theta}}_{j+1} + \Delta\boldsymbol{\theta}_{j+1} = \overline{\boldsymbol{\theta}}_j + \Delta\boldsymbol{\theta}_j + \left(\overline{\mathbf{T}}_j + \Delta\mathbf{T}_j\right)\left(\overline{\mathbf{z}}_m + \Delta\mathbf{z}_m - \overline{\mathbf{z}}_j - \Delta\mathbf{z}_j\right) \tag{5}$$

where the transformation matrix becomes,

$$\mathbf{T}_j = \overline{\mathbf{T}}_j + \Delta\mathbf{T}_j; \quad \Delta\mathbf{T}_j = \sum_{k=1}^{n} \frac{\partial \overline{\mathbf{T}}_j}{\partial z_{mk}} \Delta z_{mk} \tag{6,7}$$

In the above equations, $\overline{\mathbf{T}}_j$ denotes the transformation matrix at the parameter means, $\overline{\mathbf{T}}_j = \mathbf{T}(\overline{\boldsymbol{\theta}}_j)$, and Δz_{mk} denotes the kth element of $\Delta\mathbf{z}_m$. We seek the parameterisation, $\overline{\boldsymbol{\theta}}_{j+1} + \Delta\boldsymbol{\theta}_{j+1}$, that converges the prediction space, $\overline{\mathbf{z}}_{j+1} + \Delta\mathbf{z}_{j+1}$, upon the measurement space $\overline{\mathbf{z}}_m + \Delta\mathbf{z}_m$. Consequently $\mathbf{T}_j$ becomes a function of measured variability $\Delta\mathbf{z}_m$ according to equations (6, 7), since the updated parameters are determined at each iteration by converging the model predictions upon the measurements.

Application of the perturbation method, by separating the zeroth order and first order terms from equation (5) leads to,

$$\mathbf{O}\left(\Delta^0\right): \qquad \overline{\boldsymbol{\theta}}_{j+1} = \overline{\boldsymbol{\theta}}_j + \overline{\mathbf{T}}_j\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right) \tag{8}$$

$$\mathbf{O}(\Delta^1): \qquad \Delta\boldsymbol{\theta}_{j+1} = \Delta\boldsymbol{\theta}_j + \overline{\mathbf{T}}_j\left(\Delta\mathbf{z}_m - \Delta\mathbf{z}_j\right) + \left(\left(\sum_{k=1}^{n} \frac{\partial \overline{\mathbf{T}}_j}{\partial z_{mk}} \Delta z_{mk}\right)\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right)\right) \tag{9}$$

Equation (8) leads to the estimate of the mean of the parameters and equation (9) is used in the determination of the covariance matrix. In the well-know minimum-variance approach, the assumption of uncorrelated measurements, $\mathbf{z}_m$, and system parameters, $\boldsymbol{\theta}_j$, appeared in the 1974 paper of Collins et al. [10] and was corrected in 1989 by Friswell [11] who included this correlation after the first iteration. In this paper the effect of this omission on the converged prediction space is investigated. In this case two recursive equations having the following form for the estimation of the mean values and co-variances of the parameters are obtained as [5],

$$\overline{\boldsymbol{\theta}}_{j+1} = \overline{\boldsymbol{\theta}}_j + \overline{\mathbf{T}}_j\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right) \tag{10}$$

$$\begin{aligned}\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_{j+1},\boldsymbol{\Delta\theta}_{j+1}\right) = &\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta\theta}_j\right) - \mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T + \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_m\right)\overline{\mathbf{T}}_j^T \\ &- \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_j,\boldsymbol{\Delta\theta}_j\right) + \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_j,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T\end{aligned} \tag{11}$$

Equation (11) does not include the second order sensitivity matrix. This leads to very considerable reduction in computational effort, of great value in practical engineering applications. The transformation matrix may be expressed as the weighted pseudo inverse, which is analogous to the transformation used in deterministic model updating [1, 2]. To the zeroth order of smallness the same equation applies,

$$\overline{\mathbf{T}}_j = \left(\overline{\mathbf{S}}_j^T\mathbf{W}_1\overline{\mathbf{S}}_j + \mathbf{W}_2\right)^{-1}\overline{\mathbf{S}}_j^T\mathbf{W}_1 \tag{12}$$

In equation (12), $\overline{\mathbf{S}}_j$ denotes the sensitivity matrix at the parameter means, $\overline{\mathbf{S}}_j = \mathbf{S}\left(\overline{\boldsymbol{\theta}}_j\right)$, and the choice of $\mathbf{W}_1 = \mathbf{I}$ and $\mathbf{W}_2 = \mathbf{0}$ results in the pseudo inverse. In the case of ill-conditioned model-updating equations, the minimum-norm regularised solution is obtained when $\mathbf{W}_2 = \lambda\mathbf{I}$ and λ is the regularisation parameter that locates the corner of the L-curve obtained by plotting the norms $\left\|\overline{\boldsymbol{\theta}}_{j+1} - \overline{\boldsymbol{\theta}}_j\right\|$ *vs* $\left\|\overline{\mathbf{S}}_j\left(\overline{\boldsymbol{\theta}}_{j+1} - \overline{\boldsymbol{\theta}}_j\right) - \left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right)\right\|$ as λ is varied [12].

If the correlation between the parameters and measurements is included, then the perturbation approach leads to following four recursive equations [5],

$$\overline{\boldsymbol{\theta}}_{j+1} = \overline{\boldsymbol{\theta}}_j + \overline{\mathbf{T}}_j\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right) \tag{13}$$

$$\begin{aligned}\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_{j+1},\boldsymbol{\Delta\theta}_{j+1}\right) = &\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta\theta}_j\right) + \mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_m\right)\mathbf{A}_j^T + \mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_m\right)\overline{\mathbf{T}}_j^T - \mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T \\ &+ \left(\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_m\right)\mathbf{A}_j^T\right)^T + \mathbf{A}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_m\right)\mathbf{A}_j^T + \mathbf{A}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_m\right)\overline{\mathbf{T}}_j^T - \mathbf{A}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T \\ &+ \left(\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_m\right)\overline{\mathbf{T}}_j^T\right)^T + \left(\mathbf{A}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_m\right)\overline{\mathbf{T}}_j^T\right)^T + \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_m\right)\overline{\mathbf{T}}_j^T - \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T \\ &- \left(\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T\right)^T - \left(\mathbf{A}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T\right)^T - \left(\overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T\right)^T + \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_j,\boldsymbol{\Delta z}_j\right)\overline{\mathbf{T}}_j^T\end{aligned} \tag{14}$$

$$\mathrm{Cov}\left(\boldsymbol{\Delta\theta}_{j+1},\boldsymbol{\Delta z}_{\mathrm{m}}\right) = \mathrm{Cov}\left(\boldsymbol{\Delta\theta}_j,\boldsymbol{\Delta z}_m\right) + \left(\mathbf{A}_j + \overline{\mathbf{T}}_j\right)\mathrm{Cov}\left(\boldsymbol{\Delta z}_m,\boldsymbol{\Delta z}_m\right) - \overline{\mathbf{T}}_j\mathrm{Cov}\left(\boldsymbol{\Delta z}_j,\boldsymbol{\Delta z}_m\right) \tag{15}$$

$$\frac{\partial\overline{\boldsymbol{\theta}}_{j+1}}{\partial z_{mk}} = \frac{\partial\overline{\boldsymbol{\theta}}_j}{\partial z_{mk}} + \overline{\mathbf{T}}_j\left(\frac{\partial\overline{\mathbf{z}}_m}{\partial z_{mk}} - \frac{\partial\overline{\mathbf{z}}_j}{\partial z_{mk}}\right) + \frac{\partial\overline{\mathbf{T}}_j}{\partial z_{mk}}\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right) \tag{16}$$

where the matrix $\mathbf{A}_{j+1}$ is determined from:

$$\mathbf{A}_{j+1} = \left[\left.\frac{\partial\overline{\mathbf{T}}_{j+1}}{\partial z_{m1}}\right|_{z_{m1}=\bar{z}_{m1}}\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_{j+1}\right) \quad \left.\frac{\partial\overline{\mathbf{T}}_{j+1}}{\partial z_{m2}}\right|_{z_{m2}=\bar{z}_{m2}}\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_{j+1}\right) \quad \cdots \quad \left.\frac{\partial\overline{\mathbf{T}}_{j+1}}{\partial z_{mn}}\right|_{z_{mn}=\bar{z}_{mn}}\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_{j+1}\right)\right] \tag{17}$$

where,

$$\left.\frac{\partial \overline{\mathbf{T}}_{j+1}}{\partial z_{mk}}\right|_{z_{mk}=\bar{z}_{mk}} = \sum_{i=1}^{m} \frac{\partial \overline{\mathbf{T}}_{j+1}}{\partial \overline{\theta}_{(j+1),i}} \left.\frac{\partial \overline{\theta}_{(j+1),i}}{\partial z_{mk}}\right|_{z_{mk}=\bar{z}_{mk}} \quad ; \quad k = 1,2,\ldots,n \tag{18}$$

$$\begin{aligned}\frac{\partial \overline{\mathbf{T}}_{j+1}}{\partial \theta_{(j+1),i}} &= \left(\overline{\mathbf{S}}_{j+1}^{T}\mathbf{W}_1\overline{\mathbf{S}}_{j+1} + \mathbf{W}_2\right)^{-1} \frac{\partial \overline{\mathbf{S}}_{j+1}^{T}}{\partial \theta_{(j+1),i}} \mathbf{W}_1 \\ &- \left(\overline{\mathbf{S}}_{j+1}^{T}\mathbf{W}_1\overline{\mathbf{S}}_{j+1} + \mathbf{W}_2\right)^{-1} \left(\frac{\partial \overline{\mathbf{S}}_{j+1}^{T}}{\partial \theta_{(j+1),i}} \mathbf{W}_1 \overline{\mathbf{S}}_{j+1} + \overline{\mathbf{S}}_{j+1}^{T}\mathbf{W}_1 \frac{\partial \overline{\mathbf{S}}_{j+1}}{\partial \overline{\theta}_{(j+1),i}} \right) \left(\overline{\mathbf{S}}_{j+1}^{T}\mathbf{W}_1\overline{\mathbf{S}}_{j+1} + \mathbf{W}_2\right)^{-1} \overline{\mathbf{S}}_{j+1}^{T}\mathbf{W}_1\end{aligned} \tag{19}$$

$$\mathrm{Cov}\left(\Delta\mathbf{z}_j, \Delta\mathbf{z}_m\right) = \overline{\mathbf{S}}_j \mathrm{Cov}\left(\Delta\boldsymbol{\theta}_j, \Delta\mathbf{z}_m\right) \tag{20}$$

It should be noted that the matrix $\mathrm{Cov}\left(\Delta\boldsymbol{\theta}_j, \Delta\mathbf{z}_m\right)$ and the vector $\partial\overline{\boldsymbol{\theta}}_j / \partial z_{mk}$ are zero at the first iteration. Consequently matrix $\mathbf{A}_j$ and $\mathrm{Cov}\left(\Delta\mathbf{z}_j, \Delta\mathbf{z}_m\right)$ are zero (Equations (17), (18) and (20)). The covariance matrices $\mathrm{Cov}\left(\Delta\boldsymbol{\theta}_j, \Delta\mathbf{z}_j\right)$ and $\mathrm{Cov}\left(\Delta\mathbf{z}_j, \Delta\mathbf{z}_j\right)$ may be evaluated by forward propagation using a variety of techniques including mean-centred first-order perturbation, the asymptotic integral and Monte Carlo simulation. The above procedure and different propagation methods are described in detail by Khodaparast et al. [5].

3 Minimisation of an objective function

The deterministic finite element model updating has been considered as an optimization process in literature. Reference [13] is a good example for this class of problem. Fonseca et al. [14] recently proposed an optimisation procedure for the purpose of stochastic model updating based on the maximisation of a likelihood function [14]. As mentioned earlier, the hyperellipses represented by $\{\overline{\mathbf{z}}_m, \mathrm{Cov}(\mathbf{z}_m, \mathbf{z}_m)\}$ and $\{\overline{\mathbf{z}}_j, \mathrm{Cov}(\mathbf{z}_j, \mathbf{z}_j)\}$ define the space of measurements and predictions respectively. In order to minimise the distance and also the size difference in between these two spaces, we propose an objective function as,

$$F = \left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right)^T \mathbf{W}_1 \left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right) + w_2 \left\| \mathrm{Cov}(\mathbf{z}_m, \mathbf{z}_m) - \mathrm{Cov}(\mathbf{z}_j, \mathbf{z}_j) \right\|_F \tag{21}$$

where $\|\bullet\|_F$ is Frobenius norm, $\overline{\mathbf{z}}_m$ is estimated mean values of test results, $\mathrm{Cov}(\mathbf{z}_m, \mathbf{z}_m)$ is the covariance matrix of measured data, $\overline{\mathbf{z}}_j$ and $\mathrm{Cov}(\mathbf{z}_j, \mathbf{z}_j)$ are the estimated mean values and the covariance matrix of predictions from mathematical model at jth iteration respectively. $\overline{\mathbf{z}}_j$ and $\mathrm{Cov}(\mathbf{z}_j, \mathbf{z}_j)$ may be found by using different propagation method at each iteration. Therefore the stochastic model updating problem can be expressed as,

$$\min_{\{\overline{\theta}\}\,\{\sigma_\theta\}} \left(\left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right)^T \mathbf{W}_1 \left(\overline{\mathbf{z}}_m - \overline{\mathbf{z}}_j\right) + w_2 \left\| \mathrm{Cov}(\mathbf{z}_m, \mathbf{z}_m) - \mathrm{Cov}(\mathbf{z}_j, \mathbf{z}_j) \right\|_F \right) \tag{22}$$

subject to:

$$\{\overline{\theta}\} \succ \{\mathbf{0}\} \quad \text{and} \quad \{\sigma_\theta\} \succ \{\mathbf{0}\}$$

where $\{\overline{\theta}\}$ denotes the mean values and $\{\sigma_\theta\}$ is the standard deviations of the system parameters. The weighting matrix, $\mathbf{W}_1$, and weighting coefficient, w_2, may be chosen to make two terms in objective function as the same order. This method is not concerned with any assumption of statistical independence between the updating parameters and measurements.

4 Numerical example: 3 degree of freedom mass-spring system

The simple example shown in Figure 1 is considered, having known deterministic parameters, subject to:

$$m_i = 1.0\,\text{kg} \quad (i = 1,2,3), \quad k_i = 1.0\,\text{N/m} \quad (i = 3,4) \quad ,k_6 = 3.0\,\text{N/m}$$

and the other parameters represented as unknown Gaussian random variables with mean values and standard deviations given by,

$$\mu_{k_1} = 1.0\,\text{N/m} \quad ,\mu_{k_2} = 1.0\,\text{N/m}\,,\ \mu_{k_5} = 1.0\,\text{N/m}$$

$$\sigma_{k1} = 0.20\,\text{N/m}, \quad \sigma_{k2} = 0.20\,\text{N/m}, \quad \sigma_{k5} = 0.20\,\text{N/m}$$

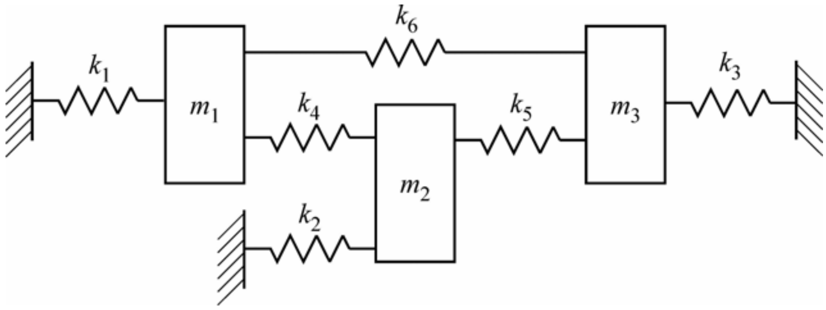

Figure 1: Case study 1: three degree-of-freedom mass-spring system.

The measured data, $\bar{\mathbf{z}}_m$ and $\text{Cov}(\mathbf{z}_m, \mathbf{z}_m)$, are obtained by using Monte Carlo simulation with 10000 samples. This number of measurements is unrealistic but is used here to demonstrate the asymptotic properties of the methods. Later the number of measurements will be varied to show the effect of this number on the parameter errors. The initial estimates of the unknown random parameters are,

$$\bar{k}_1 = \bar{k}_2 = \bar{k}_5 = 2.0\,\text{N/m} \qquad \text{Cov}(k_i) = (0.3^2)\text{N}^2/\text{m}^2 \quad i = 1,2,5$$

so that a 100% initial error in mean values and a 50% initial error in standard deviations is represented.

Results obtained by the present perturbation methods $(\mathbf{W}_1 = \mathbf{I}, \mathbf{W}_2 = \mathbf{0})$, the minimisation of an objective function $(\mathbf{W}_1 = \mathbf{I}, w_2 = 1)$, the method of Hua et al. [9], and the minimum variance estimators of Collins et al. [10] and Friswell [11] are shown in Table 1. The numbers, (1)-(6) in the table denote the following methods:

1. The proposed method in which the correlation between measured data and system parameters is omitted (Equations (10) and (11)) [5].
2. Second proposed method (minimising equation (22)). Optimization problem is solved using the Matlab Optimization Toolbox.
3. The proposed method in which the correlation between measured data and system parameters is included after first iteration (Equations (13)-(20)) [5].
4. Method introduced by Hua *et al*. [9]
5. The minimum variance method of Collins et al. [10].
6. The minimum variance method of Friswell [11].

It should be noted that the method of Hua *et al*. [9] does not require a starting estimate for the standard deviation of the unknown random parameters, which start from zero at first iteration.

Firstly, it is seen that the results obtained by method (1), when the correlation of system parameters with the measured data is omitted, are at least as good as when this correlation is included. The results achieved by method (2) are very good too. Methods (3) and (4) require the evaluation of the second-order sensitivity, which is an expensive computation and not needed when using methods (1) and (2). Finally, it is seen that the minimum variance methods (5) and (6) are really not intended for the estimation of randomised parameters to represent test-piece variability. These methods work well when the variability is limited to measurement noise from a single test piece. Figure 2 shows the convergence of the predictions upon experimental data in the space of the first three natural frequencies using method (1). 10000 samples are clearly enough to obtain an accurate estimate of the parameter variability. Figure 3 shows the convergence of the parameter standard deviations by method (1) as the number of samples is increased from 10 to 1000. In each case 10 runs of the updating algorithm were carried out to enable a range of solution errors to be determined. A different set of samples was used in each of the 10 runs. When only 10 samples were used errors were found in the range of 24-54%, while in the case of 1000 samples the errors ranged from 3-7%.

Table 2 shows the converged results and percentage errors of the parameter statistics using only 10 samples with methods (1), (2), (3) and (4). The 10 samples were different in each of the three cases, which are shown to converge to similar results. Figures 4 and 5 show the convergence of scatter of predictions upon the scatter of simulated measurements in the planes of the first and second, and second and third natural frequencies respectively. Ten measurement samples and 10000 predictions from the estimated parameter distributions by method (1) are shown.

Parameters	**Initial % Error**	**% Error (1)**	**% Error (2)**	**% Error (3)**	**% Error (4)**	**% Error (5)**	**% Error (6)**
$\bar{k}_1$	100	1.20	0.90	1.32	1.21	1.62	17.43
$\bar{k}_2$	100	-2.43	-3.07	-2.26	-2.18	-2.35	36.81
$\bar{k}_5$	100	0.71	0.84	0.57	0.23	1.86	58.20
std(k_1)	50	0.31	0.98	0.88	-0.35	-89.80	-13.36
std(k_2)	50	1.77	-0.12	0.46	-1.27	-89.85	-12.07
std(k_5)	50	1.96	-0.62	0.24	-0.16	-90.20	-58.83

Table 1: Updating results obtained by various methods (10000 samples).

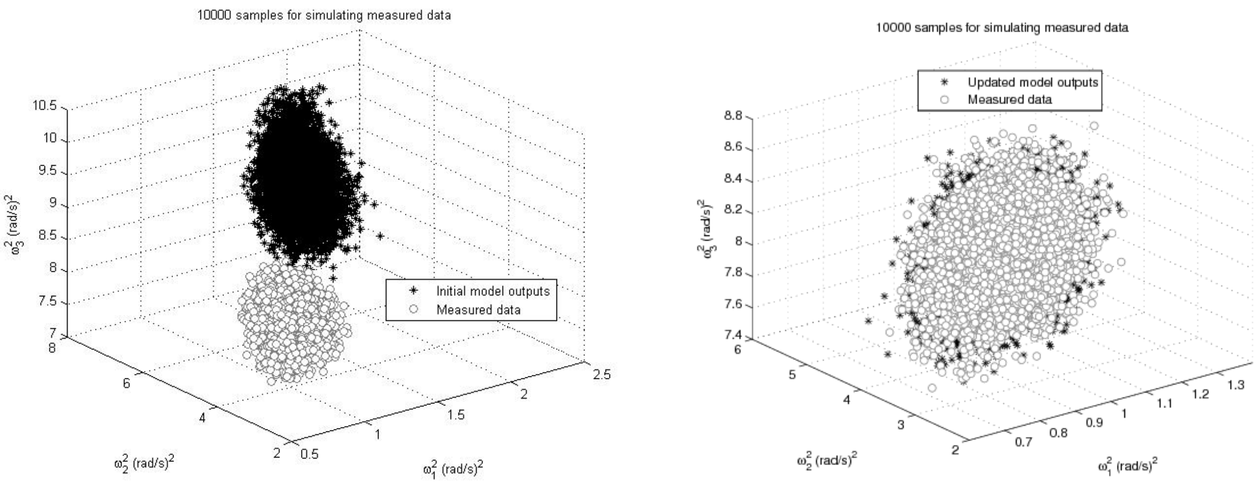

Figure 2: Initial and updated scatter of predicted and measured data: identification using method (1) with 10000 samples.

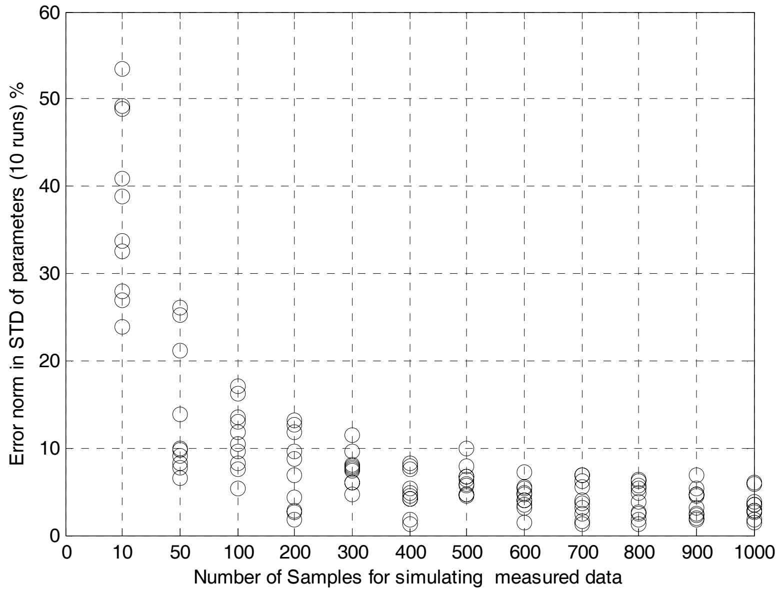

Figure 3: Error norm for parameter standard deviations using different sample sizes each with ten runs of the algorithm.

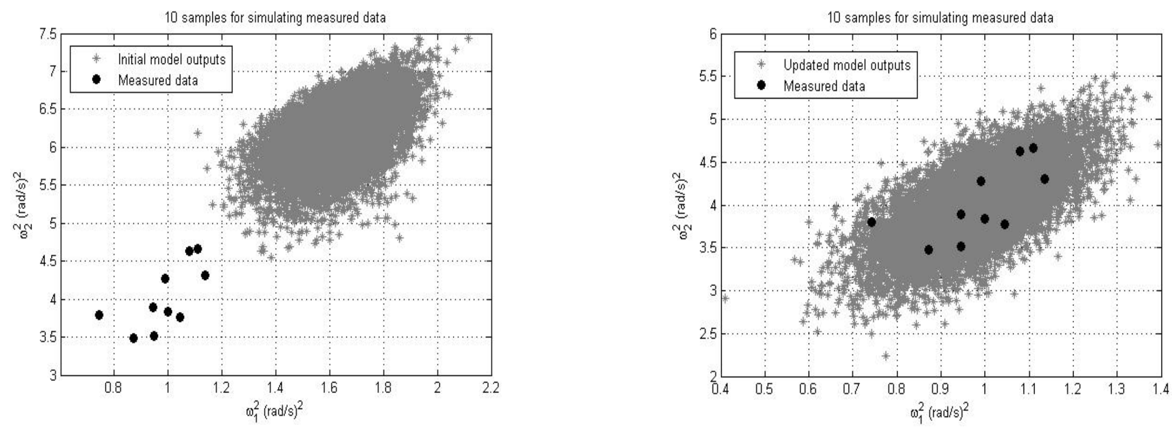

Figure 4: Initial and updated scatter of predicted data (10000 points) based upon 10 measurement samples: identification by method (1).

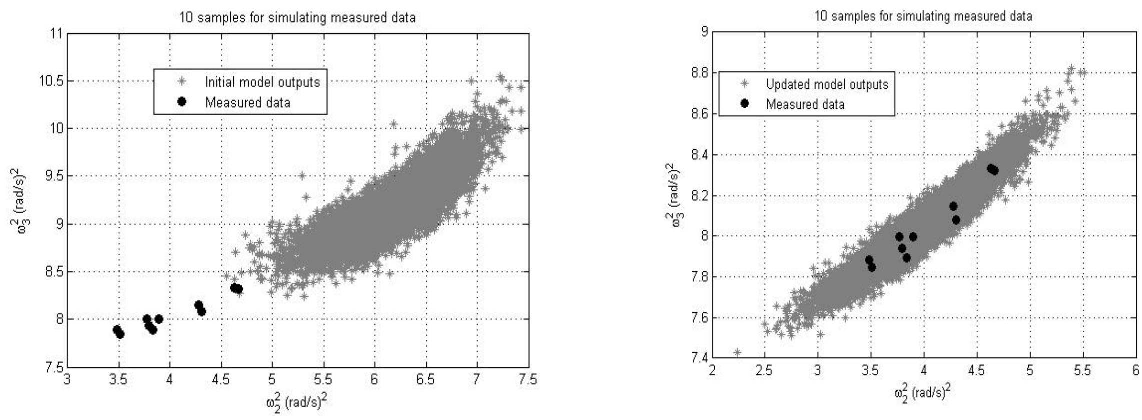

Figure 5: Initial and updated scatter of predicted data (10000 points) based upon 10 measurement samples: identification by method (1).

Parameters	Initial % Error	% Error (1)	% Error (2)	% Error (3)	% Error (4)
$\bar{k}_1$	100	4.53	-6.82	-5.42	10.81
$\bar{k}_2$	100	-8.25	7.90	-1.52	1.73
$\bar{k}_5$	100	4.21	2.00	0.69	-2.86
std(k_1)	50	-20.03	19.26	12.60	10.32
std(k_2)	50	14.35	-14.01	19.33	-7.26
std(k_5)	50	17.65	-34.28	-13.66	-36.44

Table 2: Updating results obtained by various methods (10 samples).

5 Experimental case studies

5.1 Aluminum plates with random thicknesses

Ten aluminum plates were prepared so that a contrived distribution of thicknesses, close to Gaussian, was obtained by machining. Care was taken to try to obtain a constant thickness for each plate but was not achieved perfectly. The mean value of the thicknesses was 3.975mm with a standard deviation of 0.163mm. In the experimental set up free boundary conditions were used to avoid the introduction of other uncertainties due to clamping or pinning at the edges of the plates. All ten plates had the same overall dimensions, length 0.4m and width 0.1m. A hammer test was carried out using four uniaxial fixed accelerometers. Figure 6 shows the excitation point, marked 'F', and the positions of four accelerometers, marked 'A', 'B', 'C' and 'D'. The mass of each accelerometer was 2 grams represented by lumped masses in the finite element model. The first 10 measured natural frequencies of all ten plates are given in Tables 3 and 4.

The thickness of the plates was parameterised in 4 regions as shown in Figure 7 and a finite element model was constructed consisting of 40×10 4-noded plate elements. The first six measured natural frequencies were used for stochastic model updating by method (1). A regularization parameter, $\lambda = 1 \times 10^{10}$, was found from an L curve. Figure 8 shows convergence of the mean values and COV for the 4 parameters. The initial mean and standard deviation of all 4 parameters were taken to be, $\bar{t}_i = 4$mm, $\text{std}(t_i) = 0.8$mm, $i = 1,...,4$. The initial mean value was chosen to be close to the true mean while the initial standard deviation was deliberately overestimated to represent a realistic stochastic model updating problem where little is known other than an approximation to the mean value.

The updated and measured means and standard deviations of the plate thicknesses are given in Table 5. These results are not in exact agreement but do show a considerable improvement in the thickness distributions when updated. It can be seen that the initial values of the means were chosen to be extremely close to the measured mean values. Small changes are observed in Table 5 after updating, away from the measured values obtained from averaged micrometer measurements at discrete points. The convergence of the standard deviations (shown in Table 5) from a considerable initial error is a much more significant result, demonstrating very clearly how well the method performs in converging the distribution of updating parameters upon the collection of measured thickness values. Of course the measured standard deviations are likely to be less accurate than the measured means.

The means and standard deviations of the first six measured natural frequencies were used in updating, whereas ten modes were measured in total. It is seen from Tables 6 and 7 that not only are the first six natural frequency distributions improved by updating but also the $7^{th} - 10^{th}$ natural frequency predictions (mean and standard deviations) are equally improved. This provides a good demonstration of the validity of the updated statistical model.

	Mode Number				
Plate No.	1	2	3	4	**5**
1	119.774	284.283	331.970	589.404	656.359
2	121.615	291.922	337.186	605.160	665.854
3	123.156	291.440	340.184	602.603	673.357
4	128.048	298.163	355.210	620.139	700.798
5	128.533	303.809	357.110	630.809	704.505
6	128.596	301.010	361.488	635.533	713.207
7	129.796	311.726	361.114	646.765	712.792
8	135.058	315.393	374.368	653.584	738.395
9	134.478	312.215	374.406	649.130	737.256
10	138.141	321.812	382.932	667.203	755.189
Mean	**128.720**	**303.177**	**357.597**	**630.033**	**705.771**
Std	**6.011**	**12.032**	**17.048**	**25.235**	**32.854**

Table 3: The first five measured natural frequencies (Hz) for the ten plates.

	Mode Number				
Plate No.	6	7	8	9	**10**
1	932.576	1091.603	1343.097	1628.879	1825.215
2	953.666	1106.861	1372.890	1650.395	1860.225
3	955.515	1119.445	1376.298	1669.899	1868.071
4	980.403	1165.177	1414.181	1736.714	1924.260
5	995.188	1169.660	1433.020	1743.750	1946.155
6	999.248	1184.455	1440.134	1765.415	1957.581
7	1019.052	1184.608	1467.366	1766.361	1987.556
8	1031.837	1225.375	1487.512	1825.602	2021.640
9	1023.229	1224.420	1479.268	1824.121	2013.354
10	1053.974	1253.610	1519.011	1866.665	2031.377
Mean	**994.469**	**1172.521**	**1433.278**	**1747.780**	**1943.543**
Std	**38.877**	**53.840**	**56.771**	**79.232**	**72.908**

Table 4: The 6th to 10th measured natural frequencies (Hz) for the ten plates.

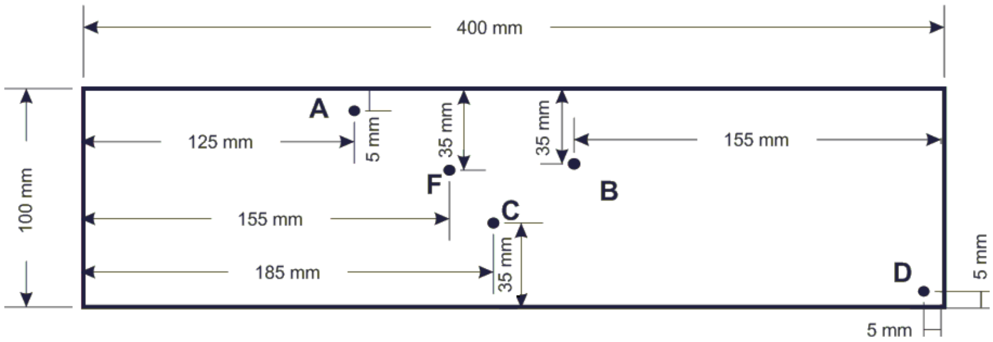

Figure 6: Arrangement of accelerometers (A, B, C, D) and excitation point (F).

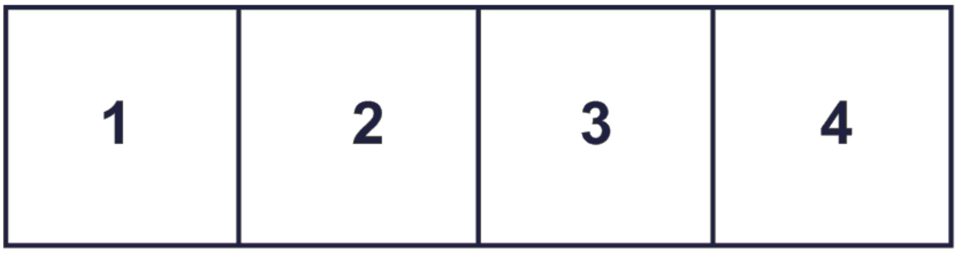

Figure 7: Parameterisation into four regions of plate thickness.

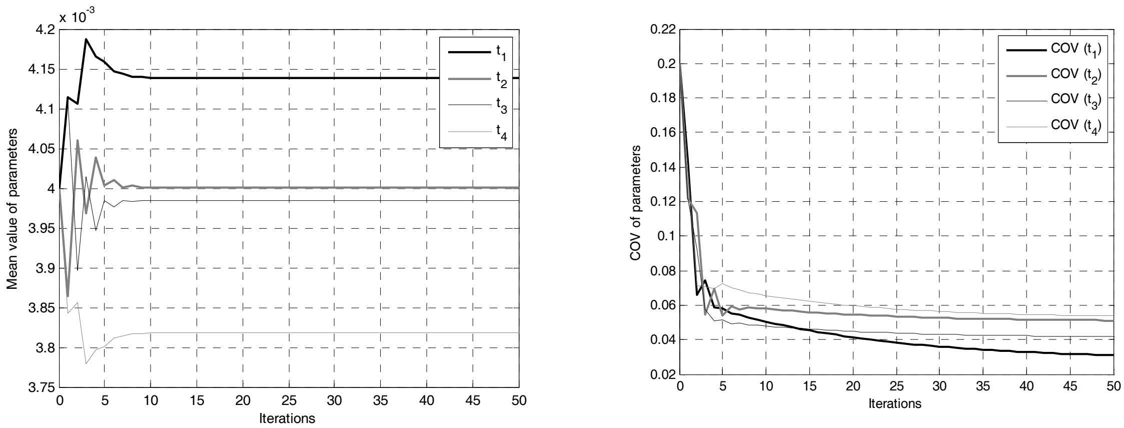

Figure 8: Convergence of parameter estimates.

	Measured Parameters	Initial Parameters	Updated Parameters	Initial FE % error	Updated FE % error
t_1 (mm)	3.978	4.000	4.140	0.553	4.072
std(t_1) (mm)	0.159	0.8	0.129	403.145	-18.868
t_2 (mm)	3.969	4.000	4.002	0.781	0.831
std(t_2) (mm)	0.161	0.8	0.204	396.894	26.708
t_3 (mm)	3.982	4.000	3.986	0.452	0.100
std(t_3) (mm)	0.164	0.8	0.166	387.805	1.219
t_4 (mm)	3.981	4.000	3.820	0.477	-4.044
std(t_4) (mm)	0.167	0.8	0.206	379.042	23.353

Table 5: Measured, initial and updated mean and standard deviation of parameters.

	Measured (Hz)	Initial FE (Hz)	Updated FE (Hz)	Initial FE % error	Updated FE % error
Mode (1)	128.720	128.321	128.111	-0.310	-0.473
Mode (2)	303.177	307.147	306.339	1.310	1.043
Mode (3)	357.597	356.645	355.185	-0.266	-0.675
Mode (4)	630.033	637.433	633.188	1.175	0.501
Mode (5)	705.771	705.467	701.777	-0.043	-0.566
Mode (6)	994.469	1002.229	996.865	0.780	0.241
Mode (7)	1172.521	1173.3951	1169.087	0.075	-0.293
Mode (8)	1433.278	1444.0183	1435.848	0.750	0.179
Mode (9)	1747.780	1748.9773	1743.491	0.069	-0.245
Mode (10)	1943.543	1952.8824	1935.851	0.481	-0.396

Table 6: Measured, initial and updated mean natural frequencies.

	Measured (Hz)	Initial FE (Hz)	Updated FE (Hz)	Initial FE % error	Updated FE % error
Mode (1)	6.011	20.943	5.750	248.411	-4.342
Mode (2)	12.032	47.385	13.777	293.825	14.503
Mode (3)	17.048	39.231	15.180	130.121	-10.957
Mode (4)	25.235	65.655	26.797	160.175	6.190
Mode (5)	32.854	71.379	28.644	117.261	-12.814
Mode (6)	38.877	108.445	40.166	178.944	3.316
Mode (7)	53.840	118.6279	46.536	120.334	-13.566
Mode (8)	56.771	148.4184	59.571	161.434	4.932
Mode (9)	79.232	177.2435	70.452	123.702	-11.081
Mode (10)	72.908	202.7527	83.427	178.094	14.428

Table 7: Measured, initial and updated std of natural frequencies.

5.2 Aluminum plates with random masses

Thirteen sets of masses having a distribution close to Gaussian were prepared. Each set included eight equal masses. The 11.5 gram set, for example, included eight masses all of 11.5 grams. The distribution of nominal masses is shown in Figure 9. The mean value of the masses was 10.063 grams with a standard deviation of 2.798 grams. Each set was glued to the surface of a plate and a hammer test was carried. The experimental set up and the positions of accelerometers and excitation points were the same as previous case study. The positions of added masses on the plate are shown in Figure 10. Each of added mass and mass of the accelerometer were represented by lumped masses in the finite element model. The first six natural frequencies of all 13 sets are given in Table 8. The second proposed method (method 2) was used in this case. As mentioned earlier, this method is an optimization problem and various optimization procedures may be used. A genetic algorithm with 20 individuals and 100 generations was used.

The first three measured natural frequencies were used for stochastic model updating by method 2. There is no need to choose initial values for mean and standard deviation of parameters in the GA algorithm but they were subjected to bounded constraints indicated in Table 9.

The identified and measured means and standard deviations of the masses are given in Table 9. As it can be seen from Table 9, the errors in identified mean and standard deviation of parameters with respect to measurements are reasonable. Obviously the identified standard deviations are less accurate than the identified means.

The means and standard deviations of the first three measured natural frequencies were used in the optimisation, whereas six modes were measured in total. It is seen from Tables 10 and 11 that, apart from the 46.8% error in the identified standard deviation of the frequency of mode 5, identified and measured means and standard deviations of natural frequencies achieved by using method 2 are in good agreement. The results show that the updated statistical model is valid.

	Mode number					
Mass (gram)	1	2	3	4	5	**6**
5.025	121.080	286.799	333.896	595.693	688.093	915.365
6.588	119.002	280.460	327.573	585.042	684.618	894.911
7.538	117.817	277.315	323.931	579.240	681.073	882.836
8.55	116.385	272.994	319.427	570.238	674.886	864.382
9.088	115.659	271.367	317.253	566.972	672.319	858.409
9.563	115.071	270.059	315.601	564.025	670.297	851.946
10.075	114.413	267.771	313.152	558.999	663.869	844.604
10.613	113.766	266.462	311.447	555.173	660.905	833.890
11.113	113.021	264.995	309.576	552.080	662.606	828.573
11.5	112.802	264.543	308.426	552.121	662.895	836.105
12.575	111.514	261.684	304.884	544.291	655.675	813.238
13.575	110.809	259.442	302.668	541.900	660.888	808.048
15.013	108.870	254.557	296.379	528.127	639.655	777.946
Mean	**114.632**	**269.111**	**314.170**	**561.069**	**667.522**	**846.943**
STD	**3.409**	**8.837**	**10.412**	**18.631**	**13.063**	**37.385**

Table 8: The first six measured natural frequencies (Hz) for a plate with 13 different sets of 8 masses attached.

	Measured Parameters	**[LB-UB]***	**Identified Parameters**	**Error %**
m (gram)	10.063	[0-20]	10.401	3.359
std (m) (gram)	2.798	[0-5]	3.278	17.155
***: LB=Lower Bound**	**UB=Upper Bound**			

Table 9: Measured, identified mean and standard deviation of parameter.

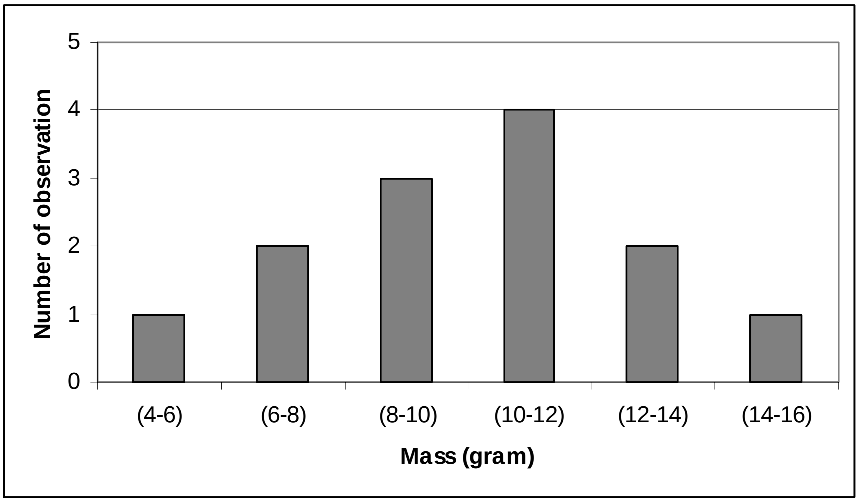

Figure (9): Distribution of masses.

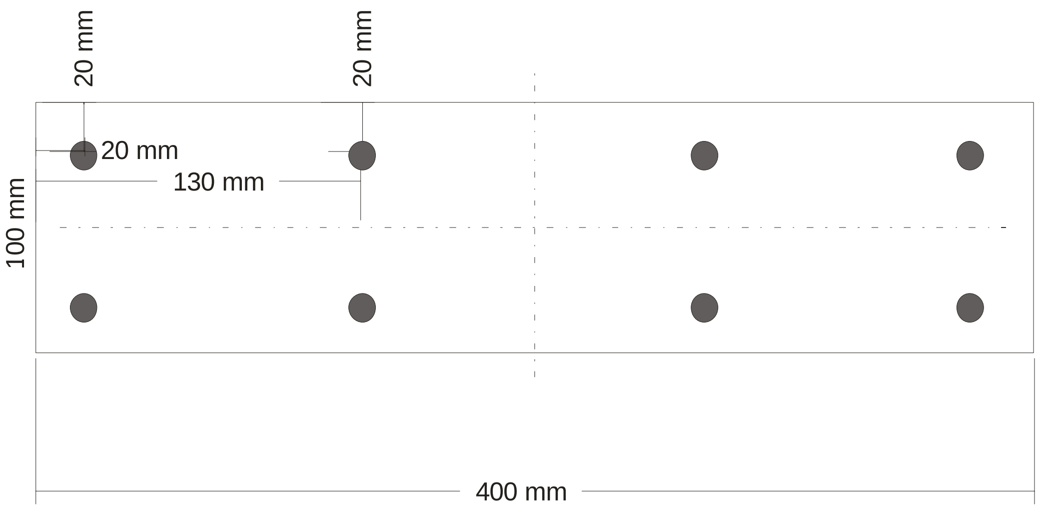

Figure (10): The positions of the masses on the plate.

	Measured (Hz)	Identified FE (Hz)	Identified FE % error
Mode (1)	114.632	113.334	-1.132
Mode (2)	269.111	270.413	0.484
Mode (3)	314.170	310.460	-1.181
Mode (4)	561.069	568.016	1.238
Mode (5)	667.522	662.697	-0.723
Mode (6)	846.943	858.850	1.406

Table 10: Measured and identified mean natural frequencies.

	Measured (Hz)	Identified FE (Hz)	Identified FE % error
Mode (1)	3.409	3.415	0.176
Mode (2)	8.837	8.568	-3.044
Mode (3)	10.412	10.562	1.441
Mode (4)	18.631	16.182	-13.145
Mode (5)	13.063	6.949	-46.804
Mode (6)	37.385	33.486	-10.429

Table 11: Measured and identified Standard deviation natural frequencies.

6 Conclusion

Two new methods based on perturbation approach and minimising an objective function to the stochastic model problem, are developed. Distributions of predicted modal responses (natural frequencies and mode shapes) are converged upon measured distributions, resulting in estimates of the first two statistical moments of the randomised updating parameters. Regularisation may be applied to the stochastic model updating equations. However both methods are computationally efficient. The two methods are demonstrated in numerical simulations and also in experiments carried out on a collection of rectangular plates with variable thickness and also variable masses on a flat plate.

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