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PARAMETER SELECTIONS FOR STOCHASTIC UNCERTAINTY IN DYNAMIC MODELS OF SIMPLE AND COMPLICATED STRUCTURES

Nurulakmar Abu Husain*, **Hamed Haddad Khodaparast and Huajiang Ouyang**

Department of Engineering
Harrison-Hughes Building
The University of Liverpool
Brownlow Hill
L69 3GH Liverpool, United Kingdom
E-mail: nurul@liv.ac.uk
E-mail: hhadad@liv.ac.uk
E-mail: h.ouyang@liv.ac.uk

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ABSTRACT

Stochastic model updating allows manufacturing variability and modelling uncertainty to be considered, so a set of analytical models with randomised parameters can be updated to match upon a set of experimental data of nominally identical test pieces. In this paper, stochastic model updating in the presence of variability in two sets of very different structures are investigated. The first set consists of nominally identical (simple) flat plates, while the second set comprises of (more complicated) formed structures. A series of experimental work is conducted on these structures and a perturbation method is employed to update their FE models to match their experimental counterparts. A Monte-Carlo propagation method is used to generate scatter plots of analytical cloud, before and after updating is performed. The main objective of this paper is to observe how updating can be adequately performed on the two sets of very different structures. Stochastic model updating is conducted with different combinations of parameters, and it is found that geometrical features (such as thickness) alone cannot converge the predicted outputs to the measured counterparts, hence material properties (for instance, Young's modulus and shear modulus) must be included in the updating process.

1. INTRODUCTION

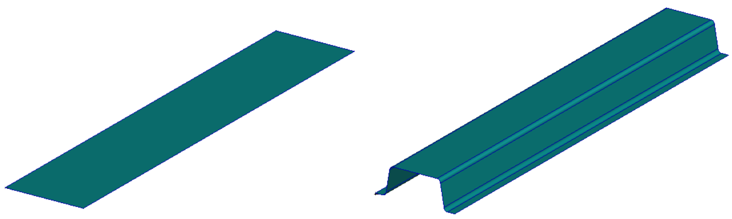
Uncertainties can be classified into *aleatory* and *epistemic* uncertainty, based on whether the source of uncertainties is reducible or not [1, 2]. Aleatory uncertainty is derived as an irreducible uncertainty that arises from heterogeneity or diversity in a population (for example, intrinsic randomness in a set of structures) and frequently cannot be reduced through further study or measurement. This type of uncertainty is also referred to as irreducible, inherent, stochastic uncertainty or variability (which is used in this paper). Epistemic uncertainty, on the other hand, represents lack of knowledge, hence can be reduced through further study or measurement. This type of uncertainty is referred to as reducible, subjective or cognitive uncertainty. Until now, there is no clear division between both types of uncertainties. For example, variability can also be a subject to lack of knowledge when information within its range is missing, which consequently becomes an epistemic uncertainty.

Demand for improved computational methods that incorporate uncertainties in numerical computation is growing. When uncertainties are taken into account, a deterministic problem then changes to a non-deterministic (or stochastic) problem. In non-deterministic problems, response of a structure cannot be precisely predicted. Therefore, the ability to numerically predict the behaviour of a structure with uncertainties is very useful and of great scientific value. Refs. [3–7] are amongst many published papers covering the stochastic model updating approach.

This paper studies how parameter selections can be sufficiently made for stochastic problems. Two sets of very different nominally identical structures are fabricated and individually tested, as described in Section 2. A stochastic model updating approach (i.e., the perturbation method presented in Ref. [6]) is used to estimate the parameter variability in the experimental modal data and the Monte Carlo method is employed to propagate the sources of variability through a deterministic FE model. The formulation of the perturbation method is presented in Section 3. Key results and contributions of the work is discussed in Section 4.

2. PLATES AND HATS: DESCRIPTIONS AND EXPERIMENTAL PROCEDURES

A set of simple rectangular plates (Fig. 1(a)) and a set of complicated hat-shaped shells (Fig. 1(b)) are used in the study. Each of the plates and hats is 564 mm long and 110 mm wide, and nominal values as tabulated in Table 1 are used for the material properties.



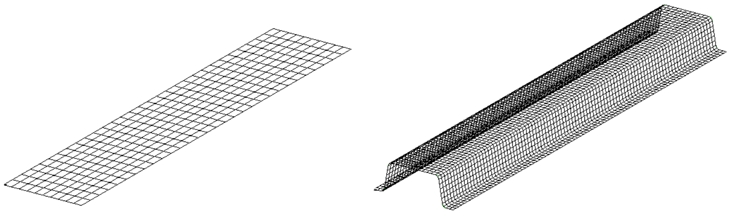
(a) Plate (b) Hat

Figure 1. Plate and hat structures

Properties	Values
Mass density (ρ)	7860 kgm^{-3}
Young's modulus (E)	210 GPa
Shear modulus (G)	83 GPa

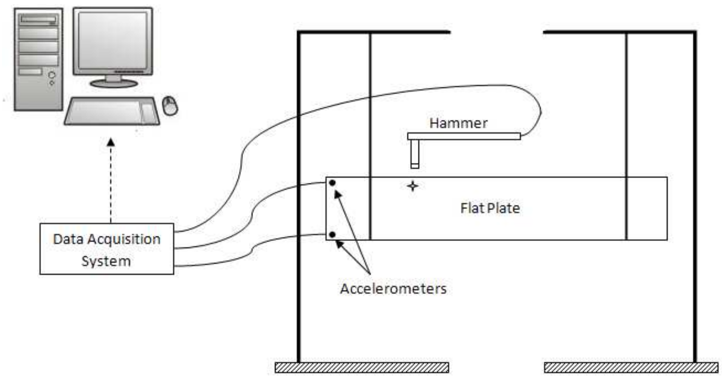
Table 1. Nominal material properties used for both sets of structures

Impact hammer modal testing [8, 9] with free-free boundary conditions was conducted, and the experimental setups for both sets of structures are shown in Fig. 3. The plates were tested using one hammer point and two measurement points as depicted in Fig. 3(a), while the hats were tested using one hammer point and five measurement points (as illustrated in Fig. 3(b)). The locations of the hammer and measurement points were chosen with care so that they are not near any nodal points. The responses were measured by using a 12-channel LMS system and extracted using an LMS PolyMAX curve-fitting procedure. The first five measured natural frequencies of the plates and the hats, together with their means and standard deviations, are given in Tables 2 and 3, respectively.

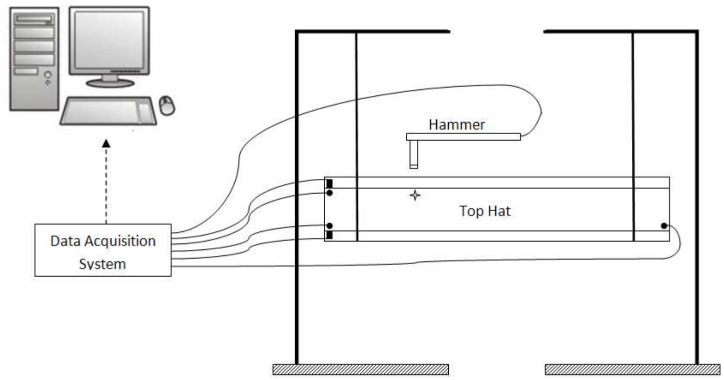


(a) FE model of the plates (b) FE model of the hats

Figure 2. FE models of the plate and hat structures



(a) Experimental setup for the plates



(b) Experimental setup for the hats

Figure 3. Experimental setups for both sets of structures

The measured outputs variances for the first five natural frequencies of the plates are

$$\sigma^2_{\boldsymbol{m}_{\text{plates}}} = [0.01 \quad 0.06 \quad 0.32 \quad 0.18 \quad 0.95]^{\mathrm{T}} \tag{1}$$

while, the measured outputs variances for the first five natural frequencies of the hats are

$$\sigma^2_{\boldsymbol{m}_{\text{hats}}} = [0.25 \quad 3.28 \quad 4.45 \quad 1.23 \quad 2.88]^{\mathrm{T}} \tag{2}$$

	Frequencies (Hz)				
Sample	1	2	3	4	5
1	24.21	67.15	78.03	132.27	159.41
2	24.12	66.93	77.98	132.11	159.34
3	24.40	67.53	78.91	133.03	160.94
4	24.12	66.88	77.80	131.88	158.94
5	24.21	67.09	77.94	132.22	159.21
6	24.32	67.44	78.57	132.78	160.51
7	24.11	66.81	77.97	131.68	159.39
8	24.11	66.88	77.74	131.81	158.85
9	24.20	67.15	78.24	132.27	159.88
10	24.20	67.08	77.86	132.23	159.25
11	24.21	67.16	77.80	132.36	159.29
12	24.24	67.16	77.91	132.36	159.56
13	24.06	66.76	77.07	131.77	157.83
14	24.07	66.86	77.07	132.00	158.06
15	24.09	66.88	77.56	131.90	158.69
16	24.04	66.81	77.27	131.86	158.37
17	24.01	66.69	77.39	131.65	158.46
18	23.96	66.60	76.24	131.43	156.49
19	24.17	67.00	77.65	132.13	158.65
20	24.23	67.17	78.20	132.46	159.78
21	24.24	67.20	78.09	132.49	159.64
22	24.26	67.23	77.78	132.57	159.10
23	23.98	66.63	77.23	131.51	158.24
24	23.96	66.62	77.16	131.47	158.26
25	24.07	66.65	77.99	131.37	158.98
26	24.22	66.97	78.30	132.05	159.48
27	24.11	66.86	77.72	131.91	158.68
28	24.02	66.73	77.17	131.73	157.95
29	24.01	66.71	77.57	131.68	158.95
30	23.94	66.50	76.95	131.31	157.45
31	24.09	66.85	76.28	131.83	156.31
32	24.00	66.72	77.39	131.64	158.46
33	24.03	66.61	77.51	131.39	158.06
Mean	24.12	66.92	77.65	131.97	158.80
Std.	0.11	0.25	0.57	0.42	0.97

Table 2. Measurement data for the plates

3. PERTURBATION METHOD FOR STOCHASTIC MODEL UPDATING

Conventional, deterministic model updating methods are based on the simple first-order Taylor series expansion and the general form of this expansion [10] is

$$\theta_{j+1} = \theta_j + \mathbf{T}_j \left(\mathbf{z_m} - \mathbf{z}_j\right) \tag{3}$$

where $\theta \in \mathbb{R}^{nx1}$ is the vector of structural parameters, $\mathbf{z_m} \in \mathbb{R}^{mx1}$ is the vector of measured data and $\mathbf{z}_j \in \mathbb{R}^{mx1}$ is the vector of predicted outputs. $\mathbf{T}_j$ is a transformation matrix, which can be

Sample	Frequencies (Hz) 1	2	3	4	5
1	69.96	271.99	286.05	333.40	393.68
2	70.43	273.79	289.01	333.93	394.85
3	70.39	273.20	287.46	334.12	393.73
4	70.49	275.59	289.46	334.44	397.46
5	70.69	270.25	287.83	335.23	399.11
6	69.02	270.34	284.12	333.04	392.57
7	69.99	273.84	288.29	333.21	394.71
8	70.21	270.27	284.62	333.39	393.76
9	70.37	274.14	286.82	334.01	395.29
10	68.97	273.78	284.47	332.40	393.46
11	69.89	272.70	285.23	332.25	393.74
12	69.41	277.18	290.48	333.83	392.97
13	70.59	274.16	287.36	334.53	397.40
14	70.21	275.16	288.40	334.92	395.95
15	70.37	275.22	289.39	334.71	396.41
16	70.62	275.67	289.83	334.92	396.72
17	69.49	275.54	288.52	335.95	394.73
18	70.65	272.88	289.26	336.59	396.71
19	70.24	273.09	285.96	335.13	397.19
20	70.53	277.06	292.79	336.09	396.92
21	70.49	274.45	288.59	335.54	398.29
22	70.26	271.75	285.52	335.24	396.26
23	69.34	273.18	286.74	335.21	395.50
24	70.48	273.10	290.39	336.28	394.07
25	69.31	274.18	287.35	334.76	392.89
26	69.92	273.44	288.19	334.40	395.93
27	70.39	274.12	289.78	334.85	397.40
28	70.59	274.69	287.01	335.33	395.99
29	70.30	275.86	292.10	336.10	396.52
30	70.14	274.44	289.09	335.26	394.36
31	69.30	272.35	286.29	335.43	393.86
32	70.11	270.31	285.63	336.24	394.18
33	70.64	274.52	289.25	335.45	396.73
Mean	70.11	273.70	287.92	334.73	395.43
Std.	0.50	1.81	2.11	1.11	1.70

Table 3. Measurement data for the hats

written as

$$\mathbf{T}_j = (\mathbf{S}_j^{\mathrm{T}}\mathbf{W}_{\varepsilon\varepsilon}\mathbf{S}_j + \mathbf{W}_{\theta\theta})^{-1}\mathbf{S}_j^{\mathrm{T}}\mathbf{W}_{\theta\theta} \tag{4}$$

with $\mathbf{W}_{\varepsilon\varepsilon}$ is a positive definite weighting matrix of the measurements, $\mathbf{W}_{\theta\theta}$ is a positive definite weighting matrix of the parameters and $\mathbf{S}_j$ is a sensitivity matrix at j^{th} iteration defined by Eq. 5 [11].

$$\mathbf{S}_j = \frac{\partial\lambda_j}{\partial\theta} = \mathbf{u}_j^{\mathrm{T}}\left[\frac{\partial\mathbf{K}}{\partial\theta} - \lambda_j\frac{\partial\mathbf{M}}{\partial\theta}\right]\mathbf{u}_j \tag{5}$$

Incorporating variability into Eq. 3 gives

$$\bar{\theta}_{j+1} + \Delta\theta_{j+1} = \bar{\theta}_j + \Delta\theta_j + (\bar{\mathbf{T}}_j + \Delta\mathbf{T}_j)\left(\bar{\mathbf{z}}_{\mathbf{m}} + \Delta\mathbf{z}_{\mathbf{m}} - \bar{\mathbf{z}}_j - \Delta\mathbf{z}_j\right) \tag{6}$$

where $\bar{\bullet}$ denotes the mean values and $\Delta\bullet$ represents the vectors of random variables. The transformation matrix is now represented by

$$\mathbf{T}_j = \bar{\mathbf{T}}_j + \Delta\mathbf{T}_j \tag{7}$$

where

$$\Delta\mathbf{T}_j = \sum_{k=1}^{n} \frac{\partial \mathbf{T}_j}{\partial z_{m_k}} \Delta z_{m_k} \tag{8}$$

Separating the zeroth-order and first-order terms from Eq. 6 gives,

$$\Delta^0\colon\ \bar{\theta}_{j+1} = \bar{\theta}_j + \bar{\mathbf{T}}_j\left(\bar{\mathbf{z}}_{\mathbf{m}} - \bar{\mathbf{z}}_j\right) \tag{9}$$

$$\Delta^1\colon\ \Delta\theta_{j+1} = \Delta\theta_j + \Delta\mathbf{T}_j\left(\Delta\mathbf{z}_{\mathbf{m}} - \Delta\mathbf{z}_j\right) \tag{10}$$

Eqs. 9 and 10 are used to determine the parameter means and the parameter covariance matrix, respectively, in the perturbation method [6]. The parameter covariance matrix equation can be written as,

$$\mathbf{C}_{\theta\theta_{j+1}} = \mathbf{C}_{\theta\theta_j} - \mathbf{C}_{\theta Z_j}\bar{\mathbf{T}}_j^{\mathrm{T}} + \bar{\mathbf{T}}_j\mathbf{C}_{EE}\bar{\mathbf{T}}_j^{\mathrm{T}} - \bar{\mathbf{T}}_j\mathbf{C}_{Z\theta_j} + \bar{\mathbf{T}}_j\mathbf{C}_{ZZ_j}\bar{\mathbf{T}}_j^{\mathrm{T}} \tag{11}$$

with $\mathbf{C}_{\theta\theta}$ is the parameters covariance matrix, $\mathbf{C}_{EE}$ is the covariance matrix of the measured outputs, $\mathbf{C}_{ZZ}$ is the covariance matrix of the predicted outputs, $\mathbf{C}_{\theta Z}$ is the covariance matrix of the parameters and the predicted outputs, and $\mathbf{C}_{Z\theta}$ the covariance matrix of the predicted outputs and the parameters, which are computed using mean-centred first order perturbation method. A significant advantage of the perturbation method [6, 12] used in this paper over another similar perturbation method by Hua et al. [7] is that only the first-order sensitivity matrix is needed in Eq. 11, hence a big reduction in terms of computational effort is achieved.

4. RESULTS AND DISCUSSION

The stochastic model updating of the plates is discussed first in this paper, followed by the updating of the hats. Problems in selecting appropriate parameters for the stochastic analysis are highlighted and discussed.

4.1 Stochastic model updating of plates

Using thicknesses as parameters

Table 2 shows the measured natural frequencies of the plates and their standard deviations. For this exercise, the stochastic model updating is performed by using only the geometrical features of the structures, i.e., the thickness of the plates. The FE model of the plates are divided into three regions, with initial value of 1.45 mm and variance of 2×10^{-4} mm^2 for each thickness. Upon convergence, the parameters vector and their corresponding variances are given by

$$\theta_{3t} = [1.4528 \quad 1.4493 \quad 1.4528]^{\mathrm{T}}, \quad \sigma^2_{3t} = 10^{-4}[1.29 \quad 0.535 \quad 1.29]^{\mathrm{T}}$$

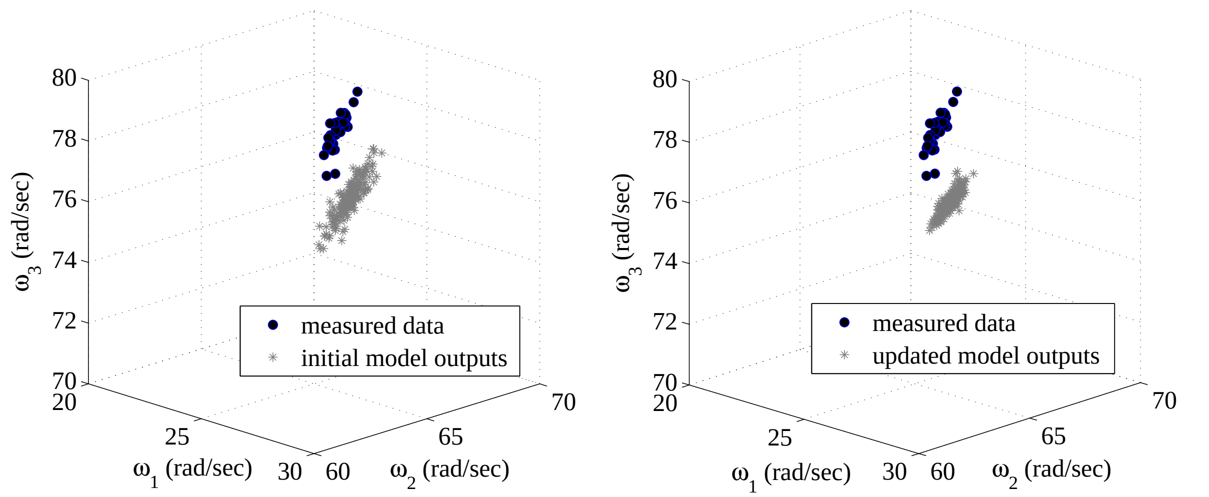
with the mean predicted outputs tabulated in Table 4 and the predicted outputs variances for the first five natural frequencies of

$$\sigma^2_{\boldsymbol{m}_{3t}} = [0.01 \quad 0.06 \quad 0.13 \quad 0.24 \quad 0.28]^{\mathrm{T}}$$

Mode	Experiment (Hz)	Initial FE (Hz)	Error (%)	Updated FE (Hz)	Error (%)
1	24.12	24.27	0.64	24.27	0.61
2	66.92	67.24	0.48	67.32	0.60
3	77.65	75.31	3.01	75.29	3.03
4	131.97	132.51	0.41	132.68	0.53
5	158.80	154.31	2.83	154.57	2.66

Table 4: Mean measured, initial and updated natural frequencies of plates using thicknesses as parameters

The error of the predicted outputs variances over the measured outputs variances is depicted in Fig. 7(a), while Fig. 4 shows the scatter plots of the measured and predicted outputs before and after the stochastic model updating process. Although the mean outputs tabulated in Table 4 indicates good agreement with the experimental data, the scatter plots illustrate that the updating procedure fails to converge the predicted outputs to the measured data.



(a) Initial scatter plot of plates using three thickness parameters

(b) Updated scatter plot of plates for using three thickness parameters

Figure 4. Initial and updated scatter plots of plates using three thickness parameters

Using material properties as parameters

If the stochastic model updating is performed using the material properties (i.e., Young's modulus and shear modulus) as the updating parameters, then the results are expected to be different. With initial parameters estimates and variances of

$$\theta_{EG} = [210\ \text{GPa} \quad 81\ \text{GPa}]^{\mathrm{T}}, \quad \sigma^2_{EG} = [4.5\ \text{GPa} \quad 0.5\ \text{GPa}]^{\mathrm{T}}$$

the predicted parameters and their variances are given as

$$\theta_{EG} = [209.6 \text{ GPa} \quad 83.8 \text{ GPa}]^T, \quad \sigma^2_{EG} = [2.6 \text{ GPa} \quad 1.5 \text{ GPa}]^T$$

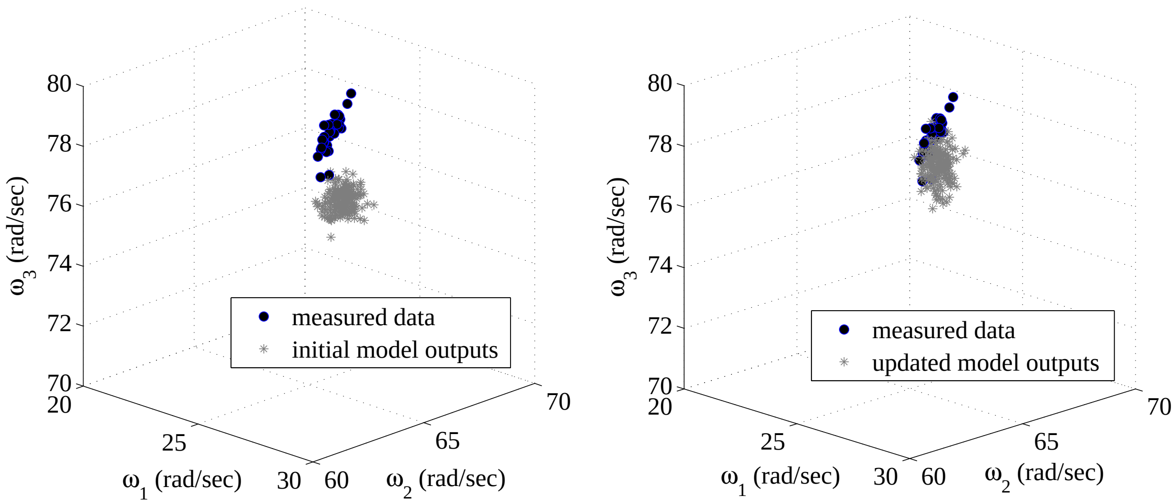
with the mean predicted natural frequencies as tabulated in Table 5 and outputs variances of

$$\sigma^2_{\boldsymbol{m}_{EG}} = [0.01 \quad 0.06 \quad 0.28 \quad 0.22 \quad 1.09]^T$$

Mode	Experiment (Hz)	Initial FE (Hz)	Error (%)	Updated FE (Hz)	Error (%)
1	24.12	24.27	0.64	24.23	0.47
2	66.92	67.24	0.48	67.04	0.18
3	77.65	75.31	3.01	76.67	1.25
4	131.97	132.51	0.41	131.93	0.03
5	158.80	154.31	2.83	156.86	1.22

Table 5: Mean measured, initial and updated natural frequencies of plates using material properties as parameters

Large error in the outputs variances of the previous exercise (i.e., using the thicknesses as parameters) is reduced significantly for the three higher modes, when the material properties are used as the updating parameters. This is shown in Fig. 7(a). Figure 5 shows the convergence of the predicted outputs over the measured outputs. Using the material properties as the updating parameters consequently converges the outputs and successfully produces reasonable parameter estimates.



(a) Initial scatter plot of plates using the material properties parameters

(b) Updated scatter plot of plates using the material properties parameters

Figure 5. Initial and updated scatter plots of plates using the material properties parameters

4.2 Stochastic model updating of hats

The findings obtained from the simple plates updating are tested when updating more complicated structures, i.e., the hats. The hats are updated firstly by using only the thicknesses as the parameters and secondly by using a combination of thickness and the material properties as the updating parameters.

Using thicknesses as parameters

The hats are divided into four regions and all of the thicknesses have the same initial values of 1.45 mm and variances of 2×10^{-4} mm^2. The predicted mean parameters and their variances are computed as

$$\theta_{4t} = [1.3989 \quad 1.4752 \quad 1.3205 \quad 1.5588]^{\mathrm{T}}, \quad \sigma_{4t}^2 = 10^{-3}[2.26 \quad 2.32 \quad 2.74 \quad 0.33]^{\mathrm{T}}$$

The mean natural frequencies of the hats using the updated parameters are closer to the measured values than the initial outputs, as can be seen from Table 6. The variances of the predicted outputs are

$$\sigma_{\boldsymbol{m}_{4t}}^2 = [0.23 \quad 3.37 \quad 2.62 \quad 1.15 \quad 2.34]^{\mathrm{T}}$$

and the difference between the predicted outputs variances over the measured outputs variances are illustrated in Fig. 7(b). The convergence of the initial and updated scatter plots are shown in Fig. 6.

Mode	Experiment (Hz)	Initial FE (Hz)	Error (%)	Updated FE (Hz)	Error (%)
1	70.11	67.28	4.03	69.12	1.41
2	273.70	256.98	6.11	268.11	2.04
3	287.92	273.47	5.02	283.29	1.61
4	334.73	334.41	0.10	333.44	0.38
5	395.43	386.35	2.30	391.66	0.95

Table 6: Mean measured, initial and updated natural frequencies of hats using thicknesses as parameters

Using combination of thickness and material properties as parameters

In this exercise, the FE model of the hats is updated by using the combination of thicknesses and material properties (i.e., Young's modulus (E)). Initial parameters estimates and variances of

$$\theta_{2tE} = [1.45\text{ mm} \quad 1.45\text{ mm} \quad 210\text{ GPa}]^{\mathrm{T}}, \quad \sigma_{2tE}^2 = [2 \times 10^{-4}\text{ mm}^2 \quad 2 \times 10^{-4}\text{ mm}^2 \quad 4.5\text{ GPa}^2]^{\mathrm{T}}$$

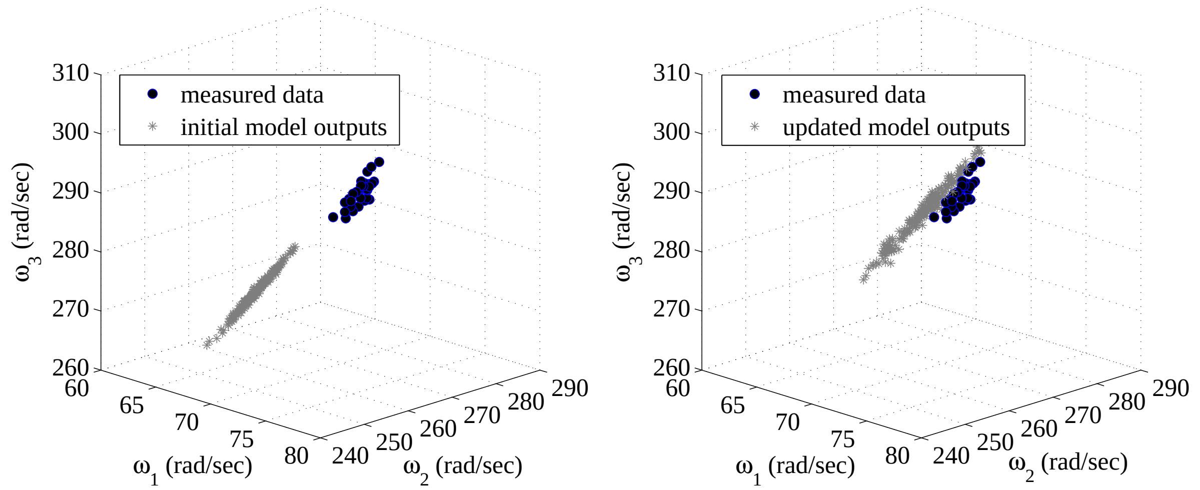
are used and the identified mean parameters and their variances are

$$\theta_{2tE} = [1.31\text{ mm} \quad 1.54\text{ mm} \quad 216\text{ GPa}]^{\mathrm{T}}, \quad \sigma_{2tE}^2 = [4 \times 10^{-3}\text{ mm}^2 \quad 1 \times 10^{-4}\text{ mm}^2 \quad 7\text{ GPa}^2]^{\mathrm{T}}$$

and the mean updated natural frequencies are very close to their measured counterparts, as tabulated in Table 7.

The predicted outputs variances are shown as follows,

$$\sigma_{\boldsymbol{m}_{2tE}}^2 = [0.26 \quad 2.18 \quad 2.28 \quad 1.24 \quad 1.94]^{\mathrm{T}}$$



(a) Initial scatter plot of hats using four thickness parameters

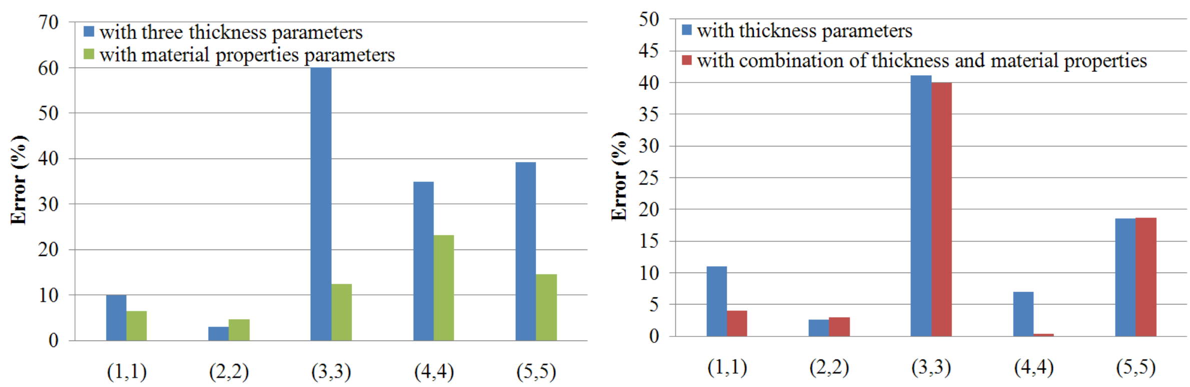
(b) Updated scatter plot of hats using four thickness parameters

Figure 6. Initial and updated scatter plots of hats using four thickness parameters

Mode	Experiment (Hz)	Initial FE (Hz)	Error (%)	Updated FE (Hz)	Error (%)
1	70.11	67.28	4.03	70.34	0.32
2	273.70	256.98	6.11	273.40	0.11
3	287.92	273.47	5.02	289.77	0.64
4	334.73	334.41	0.10	337.82	0.92
5	395.43	386.35	2.30	401.70	1.59

Table 7: Mean measured, initial and updated natural frequencies of hats using thicknesses as parameters

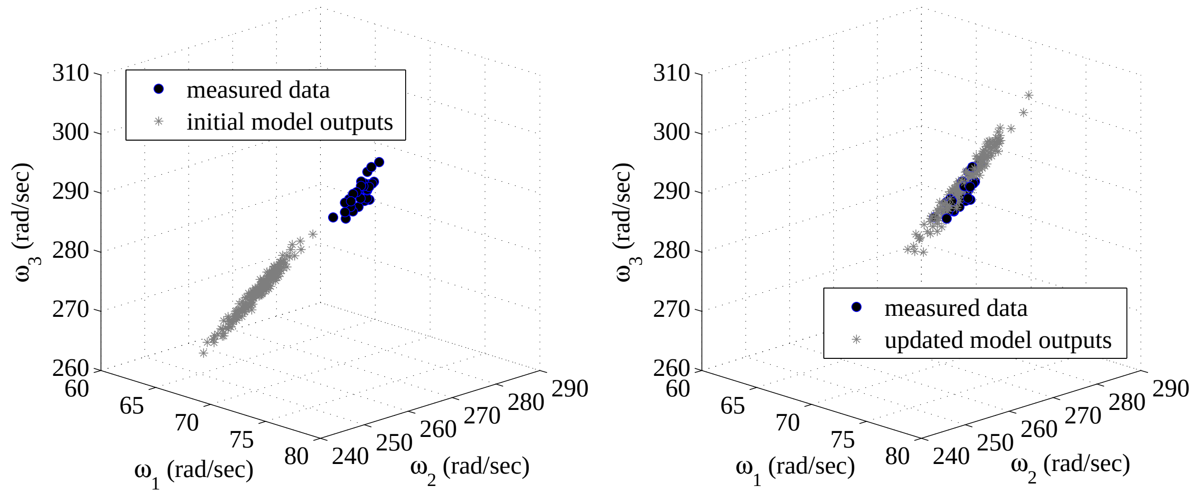
and the error of the predicted variances over the measured variances is illustrated in Fig. 7(b). The convergence of the initial and updated outputs are given in Fig. 8. It can be seen that by selecting a combination of thickness and material properties, very good convergence is obtained.



(a) Errors of the outputs variances after updating of plates

(b) Errors of the outputs variances after updating of hats

Figure 7. Errors of outputs variances after updating for both sets of structures



(a) Initial scatter plot of hats using a combination of thickness and material properties parameters

(b) Updated scatter plot of hats using a combination of thickness and material properties parameters

Figure 8: Initial and updated scatter plots of hats using a combination of thickness and material properties parameters

5. CONCLUSIONS

This paper has described parameter selections in the stochastic model updating that may be applied in future studies to quantify variability in the dynamics of structures. The study has been conducted to two very different sets of structures, i.e., simple plates and complicated hat-shaped shells, and stochastic model updating has been conducted by using different sets of parameters (i.e., the thickness and material properties). The findings indicate that selecting some of the material properties as the updating parameters provides better convergence than those updated by using only the thickness parameters.

As some general guidelines, the selection of parameters should be made by choosing the most sensitive parameters to the response of the system. This can be easily achieved by carrying out a simple sensitivity analysis. The selection of parameters should also be chosen so that the mean outputs are closer to the measured outputs, and convergence between the scatter plots of the predicted and measured outputs can be obtained. This can be achieved by including both geometrical and material properties in the updating procedure, rather than choosing a number of the geometrical properties alone, as has been demonstrated in this paper.

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