STATISTICAL PARAMETER ESTIMATION FROM MODAL DATA USING A VARIABLE TRANSFORMATION AND TWO WEIGHTING MATRICES

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ABSTRACT. Stochastic finite element model updating in structural dynamics needs statistical information on measurements and structural parameters. A stochastic model updating method based on a least squares estimator and the perturbation method is formulated. The method is capable of determining the uncertainty in structural parameters using established propagation methods such as Monte Carlo simulation and perturbation. The proposed method has been applied to the case of a simulated three degree-of-freedom mass-spring system. The results are validated by a second-order sensitivity method. The use of weighting matrices to balance errors between the two statistical indices of the estimated parameters is introduced.

KEYWORDS: Model updating, variability, uncertainty propagation, weighting matrix

1 INTRODUCTION

Conventional model updating methods use measurement information from a single structure [1, 2, 3, 12, 13] whereas stochastic model updating methods generally need multiple sets of test data from many structures built in the same way from the same materials, but with manufacturing and material variability [4, 5, 6]. This leads to improved confidence in the parameters of the updated model. Stochastic model updating may also include the case of a single test structure with varying vibration characteristics due to environmental erosion, operating loads, fatigue, wear etc. [7, 8, 9]. Observed variability in measured modal data is caused mainly by manufacturing tolerances (represented by uncertain parameters) and measurement noise.

It is clear that the use of randomised structural parameters leads to increased computation in model updating, and therefore it is important to use statistical estimation methods so that the computational effort does not become unreasonable. In this paper we consider the statistical inverse problem in which statistical information from measurements is used to identify randomized parameters. Similar to the conventional model updating problem, an initial estimate of system-parameter statistical indices must been chosen and then updated iteratively. The choice of parameters in stochastic model updating is as important as in conventional model updating and requires considerable physical insight [2, 10, 11].

In the statistical model updating of Collins *et al.* [12] and Friswell [13] the randomness arises only from the measurement noise and the updating parameters have unique values, to be found by iterative correction to the estimated means, whilst the variances are minimised. In the method described in this paper, the randomness arises from two sources, product variability (principally due to manufacturing tolerances) and measurement noise. In this case, multiple tests are carried out on nominally identical test structures each having a set of unique values for the updating parameters different from the others. Thus two spaces are defined representing the space of the measurements and the space of predictions, and our purpose is to converge the prediction space upon the space of experimental measurements. This is achieved by a least squares method which then defines a complete space of updated parameters and not just their mean values. The assumption in [12, 13] that the expected values of the parameters do not change from iteration to iteration is not appropriate in this case and the transformation matrix becomes a function of prediction variability as will be explained in the sequel. The expected value of the transformation matrix is expressed in terms of two weighting matrices which allow a balanced estimate of both the mean and standard deviation of the parameters to be achieved.

Stochastic model updating by the perturbation method [7, 8, 9, 14] needs the second-order modal sensitivities, which is time-consuming. These methods have been validated by Monte Carlo simulation. We refer to these methods as second order sensitivity methods and use them for validation in this paper. The method proposed here needs only the first-order sensitivity matrix evaluated at the expected values of estimated parameters when propagation by the perturbation method is used or by multivariate multiple regressions [5] when propagation by the Monte Carlo method is applied.

The proposed method is applied to a simulated three degree-of-freedom mass-spring system. The results obtained are shown to be in good agreement with those obtained by the second-order sensitivity method. The weighting matrices introduced are capable of balancing the errors between statistical indices of the parameters. It is shown that use of the weighting matrices can reduce the estimation errors for the standard deviation.

2 THEORY

According to the conventional, deterministic, model updating method [2], the estimate $\boldsymbol{\theta}_{j+1}$ can be updated using prior estimate $\boldsymbol{\theta}_j$ as,

$$\boldsymbol{\theta}_{j+1} = \boldsymbol{\theta}_j + \mathbf{T}(\mathbf{z}_m - \mathbf{z}_j) \tag{1}$$

where $\mathbf{z}_j$ is the vector of estimated output parameters (eigenvalues and eigenvectors), $\mathbf{z}_m$ is the vector of measured data, $\boldsymbol{\theta}$ is the vector of system parameters and $\mathbf{T}$ is a transformation matrix.

Previous authors, using the minimum variance estimator [2, 12, 13] with the constant transformation matrix **T**, have supposed that the estimated parameters are unbiased at each iteration. For the stochastic model updating presented here the assumption of unbiasedness of the parameters, $\boldsymbol{\theta}$, must be abandoned owing to the form of **T**, which becomes a function of the model variability. To begin, we define the parameters, outputs and transformation in terms of the expected value $\mathrm{E}[\bullet]$ and the variability Δ,

$$\boldsymbol{\theta} = \mathrm{E}[\boldsymbol{\theta}] + \Delta\boldsymbol{\theta} \tag{2}$$

$$\mathbf{z}_j = \mathrm{E}[\mathbf{z}_j] + \Delta\mathbf{z}_j \tag{3}$$

$$\mathbf{z}_m = \mathrm{E}[\mathbf{z}_m] + \Delta\mathbf{z}_m \tag{4}$$

$$\mathbf{T} = \mathrm{E}[\mathbf{T}] + \sum_{k=1}^{n} \mathbf{T}_{kk} \Delta z_{jk} \tag{5}$$

$\Delta\mathbf{z}_j$ denotes variability in vibration response of the mathematical model at the j^{th} iteration. This variability arises from uncertain parameters, $\boldsymbol{\theta}$, deemed responsible for the observed variability in the measured vibration data $\Delta\mathbf{z}_m$. We seek the statistics (mean and standard deviation) of these parameters that cause the convergence not only of $\mathrm{E}(\mathbf{z}_j)$ on $\mathrm{E}(\mathbf{z}_m)$ but also of $\Delta\mathbf{z}_j$ on $\Delta\mathbf{z}_m$.

We observe from equation (1) that the prediction space, defined in equation (3), should be made to converge upon the space of measured outputs, defined in equation (4). There are two measures of this convergence, namely convergence of the means, and convergence of the standard deviations. The conventional minimum variance method allows only for a single mathematical prediction, and not for the space of predictions defined in equation (3) – in this sense the minimum variance estimator provides only an incomplete statistical description. This explains why the minimum variance estimator generally produces a very good estimate of the mean, but a poor estimate of the standard deviation. A consequence of this understanding is that the transformation matrix should be different for different mathematical models within the variability $\Delta\mathbf{z}_j$. Equation (5) is then obtained by a truncated Taylor series expansion. A different matrix $\mathbf{T}_{kk}$ generally exists for every term Δz_{jk} in the vector of model variability $\Delta\mathbf{z}_j$.

Following equations may be developed from equation (1), the expectation of equation (1) and equations (2)-(5),

$$\mathrm{E}\left[\boldsymbol{\theta}_{j+1}\right] = \mathrm{E}\left[\boldsymbol{\theta}_j\right] + \mathrm{E}[\mathbf{T}]\left(\mathrm{E}[\mathbf{z}_m] - \mathrm{E}\left[\mathbf{z}_j\right]\right) + \mathrm{E}\left[O\left(\Delta^2\right)\right] \tag{6}$$

$$\Delta\boldsymbol{\theta}_{j+1} = \Delta\boldsymbol{\theta}_j + \mathrm{E}[\mathbf{T}]\left(\Delta\mathbf{z}_m - \Delta\mathbf{z}_j\right) + \left(\left(\sum_{k=1}^{n}\mathbf{T}_{kk}\Delta z_{jk}\right)\left(\mathrm{E}[\mathbf{z}_m] - \mathrm{E}\left[\mathbf{z}_j\right]\right)\right) + O(\Delta^2) \tag{7}$$

where $\mathrm{E}\left[\Delta\boldsymbol{\theta}_j\right] = \mathrm{E}\left[\Delta\mathbf{z}_j\right] = \mathrm{E}[\Delta\mathbf{z}_m] = \mathbf{0}$

Equation (6) leads to the estimate of the mean of the parameters and equation (7) is used in the determination of the covariance matrix.

A form similar to that achieved by first order perturbation [7, 8, 9, 14] may be achieved by ignoring the second-order variability terms in equation (6). We introduce an expression for the expected value of the transformation **T** that makes use of two weighting matrices, **W** and **W**$_e$,

$$\mathrm{E}[\mathbf{T}] = \mathbf{W}\,\mathrm{E}\left[\mathbf{S}_j^T\right]\left(\mathrm{E}\left[\mathbf{S}_j\right]\mathbf{W}\,\mathrm{E}\left[\mathbf{S}_j^T\right] + \mathbf{W}_e\right)^{-1} \tag{8}$$

The choice of $\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$ results in the pseudo inverse and $\mathbf{W} = \mathrm{V}_{\mathrm{j}}, \mathbf{W}_e = \mathbf{V}_e$ gives the transformation matrix defined by Collins *et al.* [12]. It will be seen that this transformation allows the estimates of the parameter means and standard deviations to be balanced, i.e., the standard deviation may be improved at the expense of the estimate of the mean and vice-versa.

The other unknown transformation matrices, $\mathbf{T}_{kk}$, $k = 1, 2, \ldots$, can be found by using a least squares estimator. Minimising the difference between measured covariance matrix and analytical covariance matrix as:

$$\min\left(\mathbf{V}_{\mathrm{m}} - \mathrm{E}\left(\Delta\mathbf{z}_j, \Delta\mathbf{z}_j\right)\right) \tag{9}$$

leads to two recursive systems of equation having the following form for the estimation of the expected value and co-variance matrix of the parameters,

$$\mathrm{E}\left[\boldsymbol{\theta}_{j+1}\right] = \mathrm{E}\left[\boldsymbol{\theta}_j\right] + \mathbf{W}\,\mathrm{E}\left[\mathbf{S}_j^T\right]\left(\mathrm{E}\left[\mathbf{S}_j\right]\mathbf{W}\,\mathrm{E}\left[\mathbf{S}_j^T\right] + \mathbf{W}_e\right)^{-1}\left(\mathrm{E}[\mathbf{z}_m] - \mathrm{E}\left[\mathbf{z}_j\right]\right) \tag{10}$$

$$
\begin{aligned}
\mathbf{V}_{j+1} = \mathbf{V}_j - \mathrm{E}\left[\Delta\boldsymbol{\theta}_j \Delta\mathbf{z}_j^T\right]\left(\mathrm{E}[\mathbf{T}]\right)^T + \mathrm{E}[\mathbf{T}]\mathbf{V}_m\left(\mathrm{E}[\mathbf{T}]\right)^T - \mathrm{E}[\mathbf{T}]\mathrm{E}\left[\Delta\mathbf{z}_j \Delta\boldsymbol{\theta}_j^T\right] \\
+ \mathrm{E}[\mathbf{T}]\mathrm{E}\left[\Delta\mathbf{z}_j \Delta\mathbf{z}_j^T\right]\left(\mathrm{E}[\mathbf{T}]\right)^T - \left(\mathrm{E}[\mathbf{T}]\mathrm{E}\left[\Delta\mathbf{z}_j \Delta\mathbf{z}_j^T\right] - \mathrm{E}\left[\Delta\boldsymbol{\theta}_j \Delta\mathbf{z}_j^T\right]\right) \\
\times \left(\mathrm{E}\left[\Delta\mathbf{z}_j \Delta\mathbf{z}_j^T\right]\right)^{-1}\left(\mathrm{E}[\mathbf{T}]\mathrm{E}\left[\Delta\mathbf{z}_j \Delta\mathbf{z}_j^T\right] - \mathrm{E}\left[\Delta\boldsymbol{\theta}_j \Delta\mathbf{z}_j^T\right]\right)^T
\end{aligned}
\tag{11}
$$

where $\mathbf{V}_j$ and $\mathbf{V}_{j+1}$ are co-variance matrix of the parameters at j^{th} and $j+1^{th}$ iterations and $E\left[\Delta\mathbf{z}_j \Delta\mathbf{z}_j^T\right]$, $E\left[\Delta\boldsymbol{\theta}_j \Delta\mathbf{z}_j^T\right]$ can be found by well established propagation techniques such as Monte Carlo simulation and the perturbation method [15]. The expected value of sensitivity matrix in equation (8) can be evaluated by multivariate multiple regression, as in [5], when using Monte Carlo methods. But Monte Carlo propagation usually needs large runs of the finite element models, which can be extremely time-consuming. Therefore we propose using propagation by a perturbation approach, which has similar results to Monte Carlo in linear cases [4]. Perturbation propagation methods need the sensitivity matrix evaluated at the expected values of the parameters $\mathrm{E}[\boldsymbol{\theta}]$. We use the following weighting matrices, from equation (8), based on the transformation matrix introduced in [12],

$$
\mathbf{W} = \mathbf{V}_j \tag{12}
$$

$$
\mathbf{W}_e = \mathbf{V}_e = \alpha^2 \mathrm{diag}\left(\mathrm{E}(\mathbf{z}_m)^2\right) \tag{13}
$$

where α is a parameter to be selected by analyst to approximately represent the level of measurement noise and may be determined, for example, from measurements on a single test structure. The weighting matrix $\mathbf{W}_e$ is important because it accounts for the difference, due to measurement noise, between the space of measurements and the prediction space as will be demonstrated later in a simulated example.

The variability in measured data arises from two sources, namely measurement noise and model variability due to uncertain parameters. Therefore the co-variance of measured data may therefore be written in the following form:

$$
\mathbf{V}_m = \mathbf{V}_u + \mathbf{V}_e \tag{14}
$$

where $\mathbf{V}_m$ is the measurement co-variance matrix, $\mathbf{V}_u$ is co-variance matrix arising from model uncertainty and $\mathbf{V}_e$ is the co-variance matrix due to measurement noise. The measurement noise depends on the experimental equipment, the test environment and data processing. Therefore modal variability due to uncertain parameters and measurement noise are statistically independent.

3 SIMULATED EXAMPLE

The proposed method was applied to the simple three degree-of-freedom mass-spring system shown in Figure 1. The nominal values for the simulated experimental system are chosen to be the same as in [5],

$$
m_i = 1.0\,kg \quad (i = 1,2,3), \quad k_i = 1.0\,N/m \quad (i = 1,\ldots,5) \quad, k_6 = 3.0\,N/m
$$

$$
\sigma_{k1} = 0.20\,N/m, \quad \sigma_{k2} = 0.20\,N/m, \quad \sigma_{k5} = 0.20\,N/m
$$

where σ_{k1}, σ_{k2} and σ_{k5} are nominal standard deviation of three uncertain parameters k_1, k_2 and k_5. Simulated co-variances of the experimental modal data due to uncertain parameters ($\mathbf{V}_u$ in equation (14)) can be found (e.g. [16]),

$$\mathbf{V}_u = \mathbf{S}_n \mathbf{V}_n \mathbf{S}_n^T \tag{15}$$

where $\mathbf{S}_n$ is sensitivity matrix evaluated at nominal expected values of the parameters and $\mathbf{V}_n$ is nominal co-variance matrix of the parameters. The erroneous random parameters are assumed as follow,

$$\mathrm{E}(k_1) = \mathrm{E}(k_2) = \mathrm{E}(k_5) = 2.0\, N/m \qquad \sigma_{k1} = 0.3,\ \sigma_{k2} = 0.3, \sigma_{k5} = 0.3$$

Indeed, the measured covariance matrix ($\mathbf{V}_m$ in equation (14)) is much more significant than measurement noise ($\mathbf{V}_e$ in equation (14)) in the presence of variability in measured data due to manufacturing tolerances, damage and wear etc. In order to have a good estimation of measurement noise, the following indicator is defined,

$$r_e = \frac{\|\mathbf{V}_e\|}{\|\mathbf{V}_m\|} = \frac{\|\mathbf{V}_e\|}{\|\mathbf{V}_u + \mathbf{V}_e\|} = \frac{\left\|\alpha^2 \mathrm{diag}\left(\mathrm{E}(\mathbf{z}_m)^2\right)\right\|}{\left\|\mathbf{S}_n \mathbf{V}_n \mathbf{S}_n^T + \alpha^2 \mathrm{diag}\left(\mathrm{E}(\mathbf{z}_m)^2\right)\right\|} \tag{16}$$

where $\|\ \|$ denote the norm of the matrix. This indicator shows how much of the space created by variability in measured data is occupied by measurement noise. Now we demonstrate the proposed method with a plausible simulation. In this case we assume that there is an exactly-simulated fully-populated covariance matrix of measured model variability according to equation (15). Three cases of $r_e = 0\%$, $r_e = 10\%$ and $r_e = 30\%$ are considered. We expect that the method should be capable of regenerating the exact values of simulated parameters when $r_e = 0\%$. Table 1 shows this to be the case and the second-order sensitivity based method verifies the results. Convergence of the Expected value and standard deviation of the uncertain parameters by using proposed method is shown in Figure 2. In the cases of $r_e = 10\%$ and $r_e = 30\%$, results have were obtained when (a) the weighting matrices were neglected ($\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$) and (b) when weighting matrix were determined according to equations (12) and (13). As can be seen in Table 2, the weighting matrices can be used to balance the errors in the estimation of both statistical indices of updated parameters. In other words using the weighting matrices can reduce the errors in estimated standard deviation while increasing the errors in the estimation of the expected value. Figures 3-6 show of the expected value and standard deviation of the uncertain parameters converge for the cases of $r_e = 10\%$ and $r_e = 30\%$.

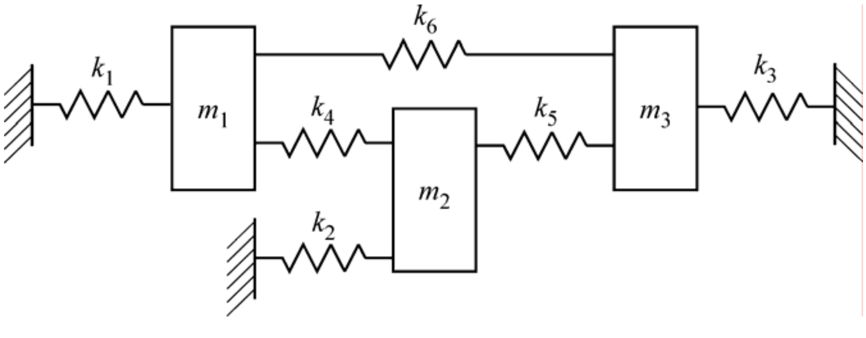

Figure 1. Three degree-of-freedom mass-spring system [5]

Parameter	Initial Error (%)	Error after updating by PM % $r_e = 0\%$	Error after updating by SSM % $r_e = 0\%$
$E(k_1)$	100	0.00	0.00
$E(k_2)$	100	0.00	0.00
$E(k_5)$	100	0.00	0.00
σ_{k_1}	50	0.00	0.00
σ_{k_2}	50	0.00	0.00
σ_{k_5}	50	0.00	0.00

Table 1. Results by the proposed method (PM) and the second-order sensitivity method (SSM) in an ideal case.

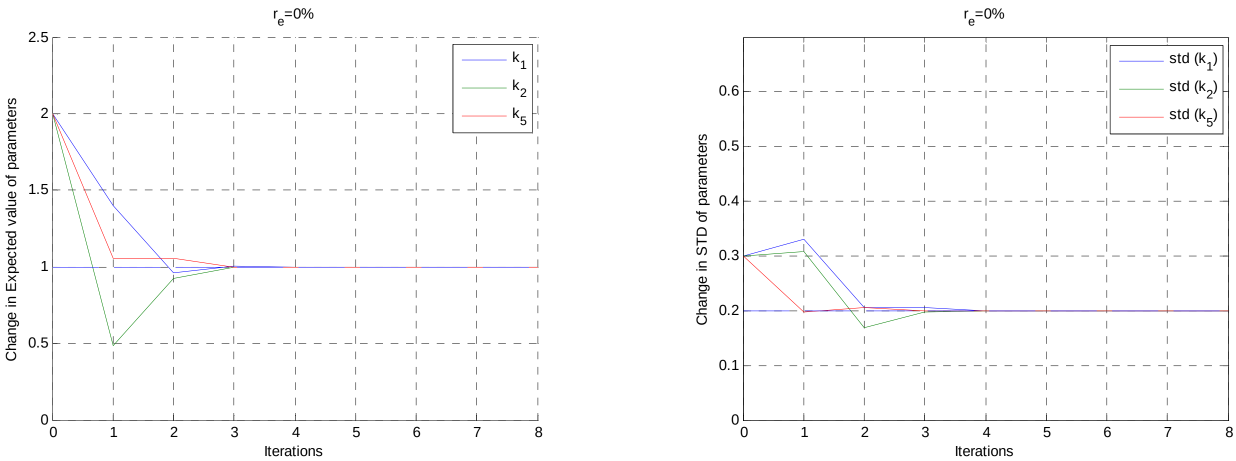

Figure 2. Convergence of parameter estimates in proposed method- $r_e = 0\%$

Parameter	%Error after updating $\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$ $r_e = 10\%$	%Error after updating $\mathbf{W} = \mathbf{V}_j, \mathbf{W}_e = \mathbf{V}_e$ $r_e = 10\%$	%Error after updating $\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$ $r_e = 30\%$	%Error after updating $\mathbf{W} = \mathbf{V}_j, \mathbf{W}_e = \mathbf{V}_e$ $r_e = 30\%$
k_1	0.00	11.90	0.00	18.75
k_2	0.00	-11.32	0.00	17.22
k_5	0.00	3.32	0.00	4.87
σ_{k_1}	31.44	-24.86	83.2995	-28.16
σ_{k_2}	31.96	-23.24	84.5194	24.87
σ_{k_5}	4.91	-7.55	15.3439	11.63

Table 2. Errors after updating by the proposed method

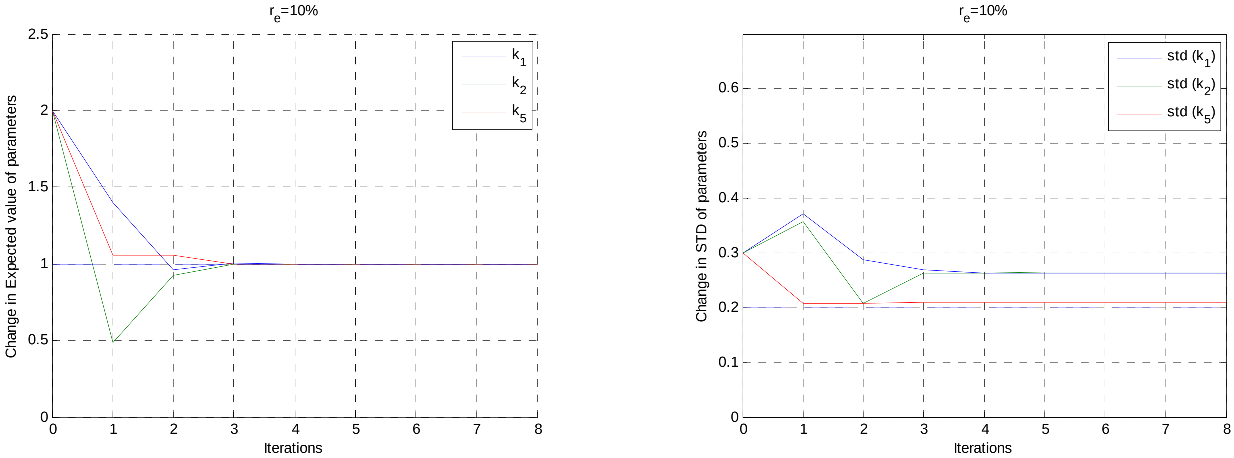

Figure 3. Convergence of parameter estimates in proposed method- $r_e = 10\%$ - $\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$

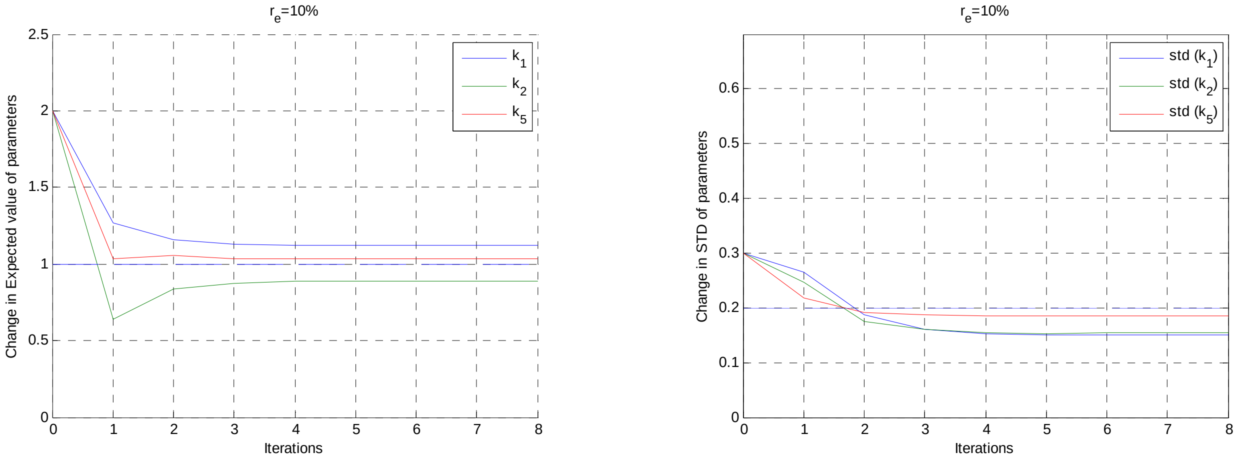

Figure 4. Convergence of parameter estimates in proposed method - $r_e = 10\%$ - $\mathbf{W} = \mathbf{V}_j, \mathbf{W}_e = \mathbf{V}_e$

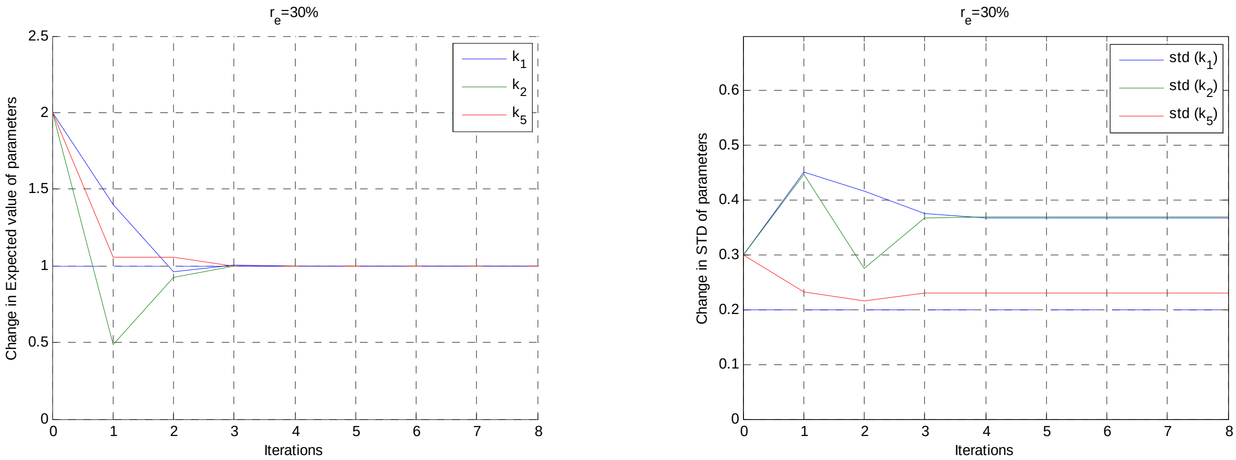

Figure 5. Convergence of parameter estimates in proposed method- $r_e = 30\%$ - $\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$

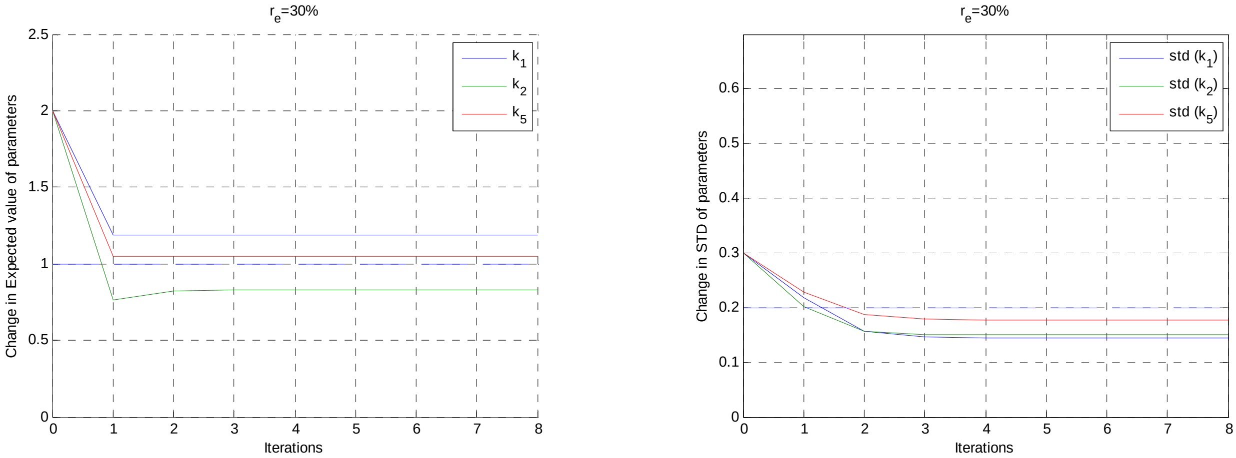

Figure 6. Convergence of parameter estimates in proposed method - $r_e = 30\%$ - $\mathbf{W} = \mathbf{V}_j$, $\mathbf{W}_e = \mathbf{V}_e$

The effect of the weighting matrices may be understood from Figure 7, which shows the error norm of the expected parameter values and the standard deviations versus the measurement noise indicator r_e. Two cases are considered, $\mathbf{W} = \mathbf{I}, \mathbf{W}_e = \mathbf{0}$ and the weighting matrices determined by equations (12)-(13). Figure 7 shows that the latter case leads to the better estimation of the standard deviation when the measurement noise is significant.

Another important limitation in practical work is the effect of errors in measured covariance matrix due to uncertain parameters ($\mathbf{V}_u$ in equation (14)). In practice there are likely to be errors in this matrix due to the scarcity of data from nominally identical test pieces manufactured within tolerances. Therefore as an example of a practical limitation, a matrix achieved by equation (15) containing 20% error is considered for the simulated co-variance matrix $\mathbf{V}_u$. Figure 8 shows error norm of the estimated parameters versus measurement noise indicator r_e. The effective weighting matrix can improve the error caused by inaccurate measurement of $\mathbf{V}_u$. Although the estimation of the expected value of the parameters are less accurate when using the weighting matrices (left diagram in Figure 8), the right diagram shows that the estimation of the standard deviation is improved. In particular the considerable errors in the estimated standard deviation caused by model variability and measurement noise can be reduced by applying the weighting matrices. For instance if we have1.4% measurement noise ($\alpha = 0.014$) which leads to $r_e = 10\%$, the error norm increases in the mean value by 15.1% (blue curve instead of zero (red line)) but decreases in the standard deviation (24.4 % (blue line) instead of 57.2 % (red line). The standard deviation is calculated with a very significant error when the weighting matrices are not used, as can be seen from the red line in the right-hand diagram of Figure 8.

Figure (9) shows convergence of the prediction space upon the space of simulated experimental measurements. This space may define by variation of system-eigenvalues within a normal distribution function. This convergence is achieved in a case of 1.4 % measurement noise by using the weighting matrices defined in equations (12)-(13).

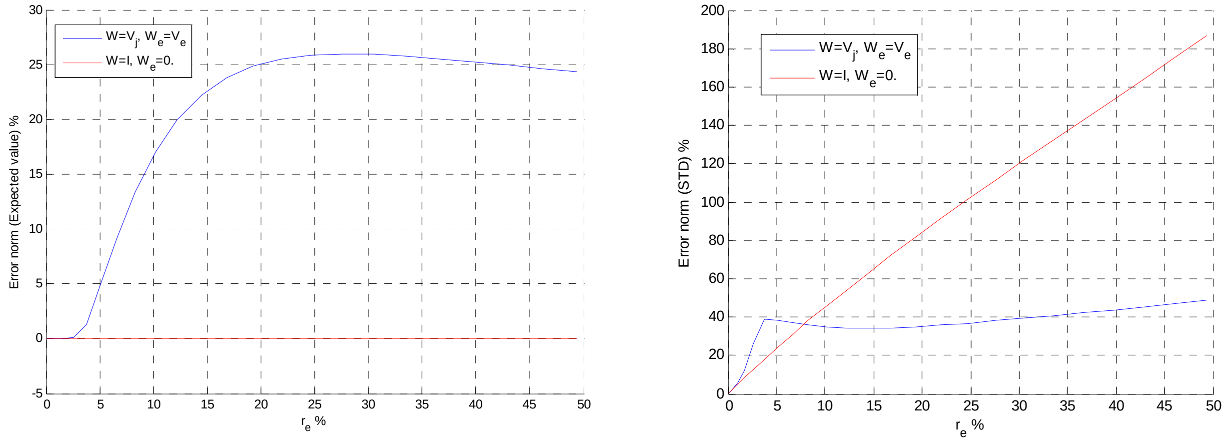

Figure 7. Error norm in expected value and STD vs. measurement noise indicator

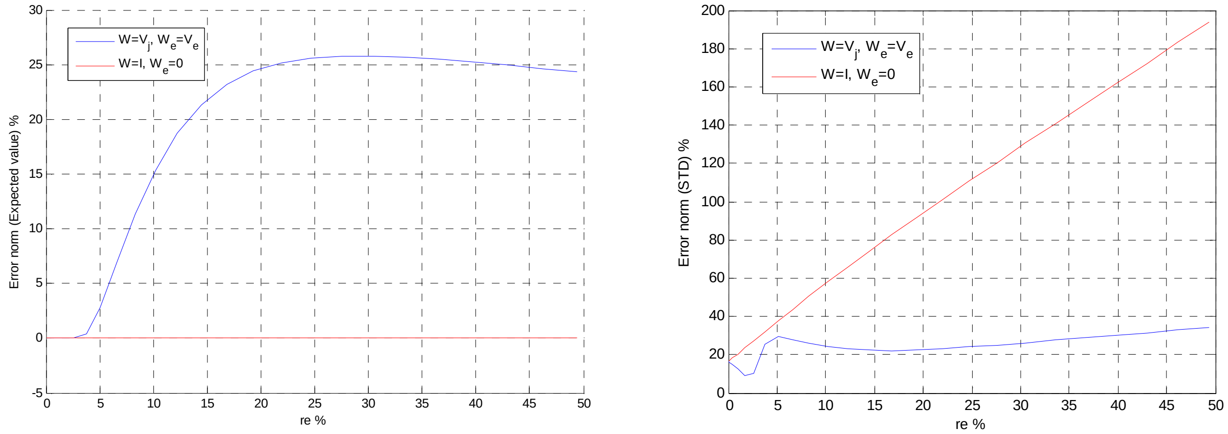

Figure 8. Error norm in expected value and STD vs. measurement noise indicator

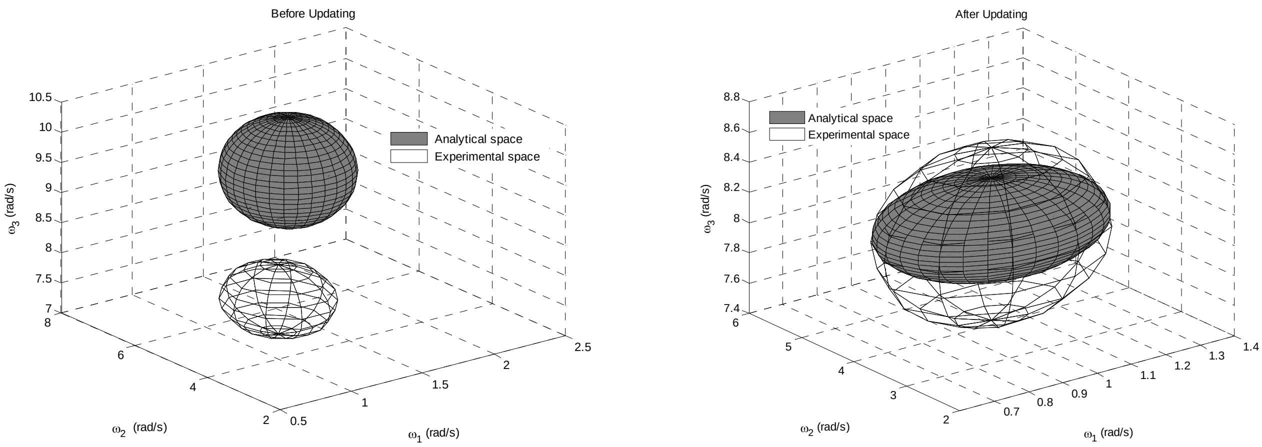

Figure 9. Convergence of analytical space upon measured space.

4 CONCLUSION

A method is developed for stochastic model updating using statistical indices and a perturbation approach with first-order sensitivities. Monte-Carlo and perturbation methods for uncertainty propagation may be applied. The method makes us of a variable transformation matrix to converge the space of model predictions upon the space of measured modal data. Weighting matrices are used to balance the errors between the means and standard deviations of the estimated structural parameters. The method is validated by using a second-order sensitivity approach and simulated examples with a three degree-of-freedom mass-spring system provide a demonstration of how the technique could be applied in practice.

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