The Role of the Kutta-Joukowski Condition in the Numerical Solution of Euler Equations for a Symmetrical Airfoil.

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Main Points

▷ Lifting Flows for for an airfoil in the framework of an inviscid compressible ¤ow.

▷ How Euler codes based on FVM and FDM deal with the Kutta-Joukowski condition.

▷ Weak Implementation and Strong implementation of the Kutta-Joukowski CONDITION.

▷ Generation of circulation when time marching Euler calculations are performed with the Kutta-Joukowski condition imposed at the sharp trailing edge.

▷ Discussion of the vorticity production and distribution ion the computational domain in the limelight of grid re£nement study.

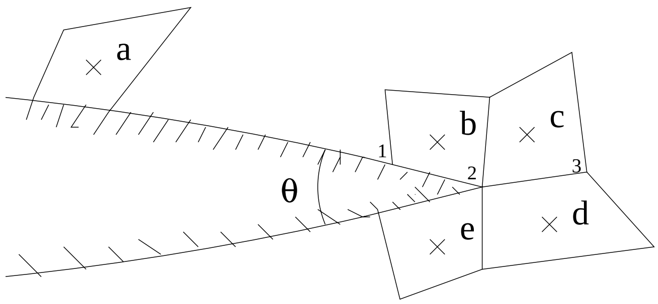
The Classical Theory of Lift

▷ A cylinder of radius R, spinning about its axis in an anti-clockwise direction, in an inviscid incompressible Flow of free-stream velocity $U_\infty \mathbf{i}$.
The stream function for the lifting Flow over the cylinder is given by

$$\Psi \;=\; U_\infty \; r \; sin\theta \Big(1 \;-\; \frac{R^2}{r^2}\Big) \;+\; \frac{\Gamma}{2\,\pi} \; ln\Big(\frac{r}{R}\Big)$$

▷Airfoil with a sharp trailing edge.
▷ Solution is dependent on the value of Γ and is therefore non unique.
▷In£nite number of solutions.
▷ The relevant one is given by the Kutta-Joukowski hypothesis
$The\ Flow\ should\ leave\ the\ trailing\ edge\ smoothly$
▷ This solution is then termed as the relevant Euler solution
▷ It is the solution of the Naviers-Stokes Equations in the limit of viscosity $\mu \;\rightarrow\; 0$ but $\mu \;\neq 0$.

Euler Codes



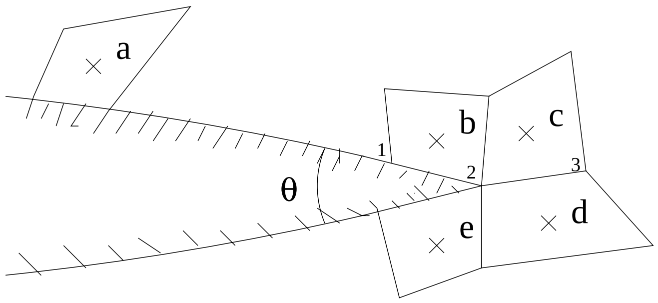
$$\mathbf{q_{te}} \cdot \mathbf{n_u} = 0 \qquad \mathbf{q_{te}} \cdot \mathbf{n_l} = 0$$

The Boundary Condition at the Kutta point is therefore $\mathbf{q_{te}} = 0$
▷Cell-Vertex Finite Volume (CVFVM)and Cell-Centred Finite Volume (CFVM)
(i) Codes based on First order accurate computations
(ii) Linear reconstruction (to enhance order of accuracy)
The cell-averages at the centroids of the cells are updated
For the cell with centroid b:

$$\mathbf{flux}_{[\mathbf{1},\mathbf{2}]} \;=\; \mathbf{incident\ flux}(\mathbf{same\ as\ that\ at\ b}) \\ + \mathbf{reflectedflux}(\mathbf{wall\ boundary\ condition})$$

Linear Reconstruction
The Flow variables are assumed to vary linearly within a cell.
The gradient of the Flow variables at centroid b are computed.

$$f(x,y) = f(x_b, y_b) + (x - x_b)\left(\frac{\partial f}{\partial x}\right)_b + (y - y_b)\left(\frac{\partial f}{\partial y}\right)_b$$

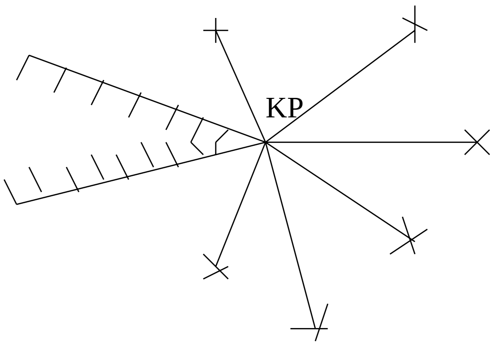


▷These methods do not take into account the fact that the solid boundary condition suddenly changes from $\mathbf{q_{te}} \cdot \mathbf{n_u} = 0$ to $\mathbf{q_{te}} \cdot \mathbf{n_l} = 0$ at the Kutta point.

The Flux on the cell face (2,3) is the same in the state update of centroids c and d.

Thus, the Flow will be prevented from sharply turning around the Kutta Point thus satisfying the Kutta Joukowski condition.

▷Finite Difference based Method (FDM)



▷The necessity of updating the state variables at the Kutta point comes into sharper focus

▷ We obtain the space derivatives at the Kutta point KP in terms of the neighbouring data.

▷Given the Flow variables ρ_{kp}^{n} , u_{kp}^{n} , v_{kp}^{n} , $\mathbf{p_{kp}^{n}}$ at time level n at the Kutta point, how to determine ρ_{kp}^{n+1} , u_{kp}^{n+1} , v_{kp}^{n+1} , $\mathbf{p_{kp}^{n+1}}$ using the data in the connectivity of the Kutta point.

▷The solution of the partial differential equation of the Flow, together with the prevailing boundary condition, is obtained.

(i)**Weak implementation** of the boundary condition at the Kutta point
example : The update of the state variables at the Kutta point is indirrectly addressed (CCFVM)

(ii)**Strong implementation** of the boundary condition at the Kutta point
The Kutta Joukowski condition is imposed and then pressure and density are updated using data at the nodes in the connectivity of the Kutta point.

Vorticity production due to the baroclinic mechanism

▷How do Euler codes applied to computations of the compressible inviscid ¤ows generate vorticity and produce enough circulation?

▷ Vorticity production in viscous ¤ows

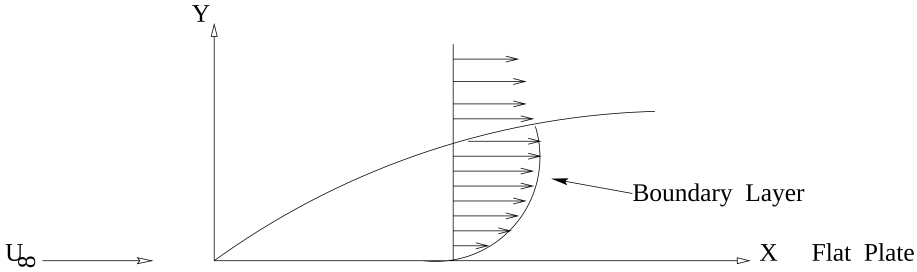


Figure 1: Viscous Flow over a Flat Plate

▷Large velocity gradient , $\dfrac{\partial u}{\partial y}$

Velocity changes sharply from its no slip value of zero to non zero value at the edge of the layer

Vorticity is produced near the ¤at plate and convected.

▷ Euler codes used for computing lifting ¤ows around the airfoil

▷ The Kutta-Joukowski condition can be regarded as a one point no-slip boundary condition $\mathbf{q_{t.e.}} = 0$ (for $\theta = 0$) at the Kutta point.

production of vorticity and Boundary Condition

at the Kutta point

The sudden change in boundary condition causes vorticity production through the $baroclinic\ mechanism$.(Balasubramaniam *et al*, CFD Centre, IISc., Bangalore)
The momentum equation for an inviscid compressible Flow.

$$\frac{\partial\, \mathbf{q}}{\partial\, t} \,+\, \mathbf{q}.\nabla\mathbf{q} \,=\, -\frac{1}{\rho}\,\nabla p$$

$$\frac{\partial\, \omega}{\partial\, t} \,+ u_1\frac{\partial\omega}{\partial x} \,+\, u_2\frac{\partial\omega}{\partial y} \,=\, -\frac{1}{\rho^2}\left[\nabla\mathbf{p}\,\wedge\,\nabla(\rho)\right]$$

▷ If $\nabla\mathbf{p}$ is not parallel to $\nabla\rho$, then the term on the rhs produces vorticity.
▷For inviscid Flows with the Kutta-Joukowski condition prevailing at the trailing edge, there is a sudden discontinuity in the wall boundary condition.
▷Large velocity gradients develop in that region and hence Flow becomes non-isentropic there

$\triangleright$the lift experienced by an airfoil in a subsonic Flow at a given angle of attack is a consequence of the circulation around the airfoil.

$$\Gamma = - \oint_c \mathbf{q}.ds = - \int\int_s \omega \, . \, dS$$

, where $\omega = \nabla \wedge \mathbf{q}$

Numerical Results

▷using the q-LSKUM (The Least Squares Kinetic Upwind Method based on Entropy variables)

▷£nite difference based scheme to solve the 2d Euler Equations

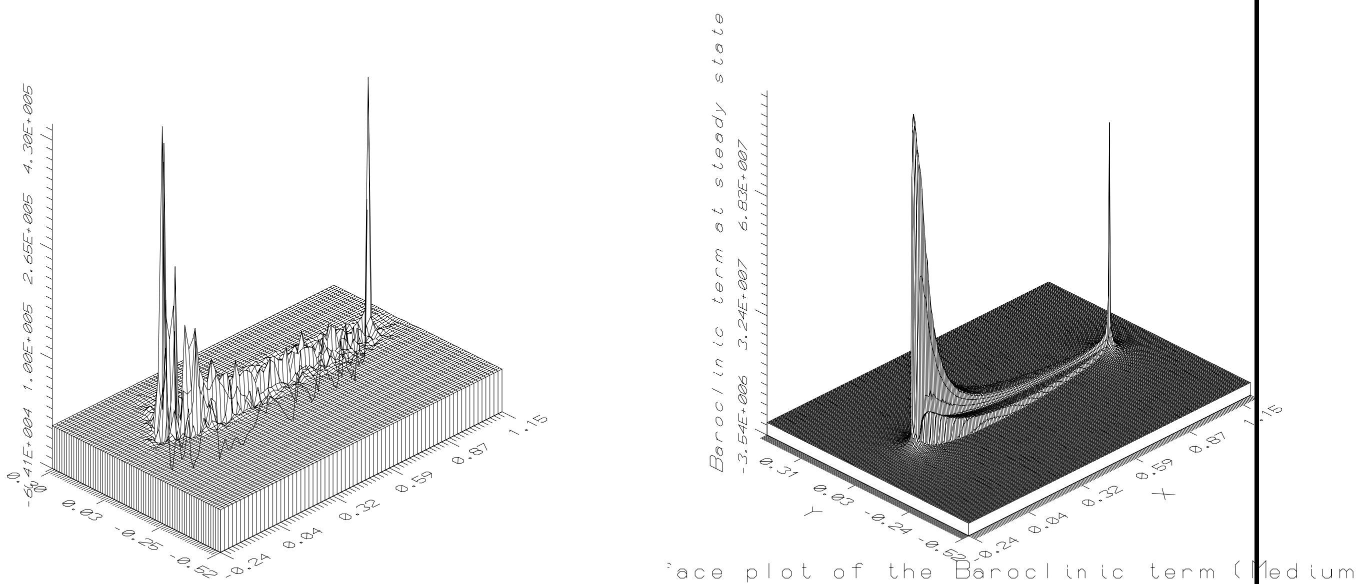


Figure 2: Surface Plots of the Baroclinic term, at steady state, a coarse and a medium grid respectively ($M_\infty = 0.63, \alpha = 2^\circ$)

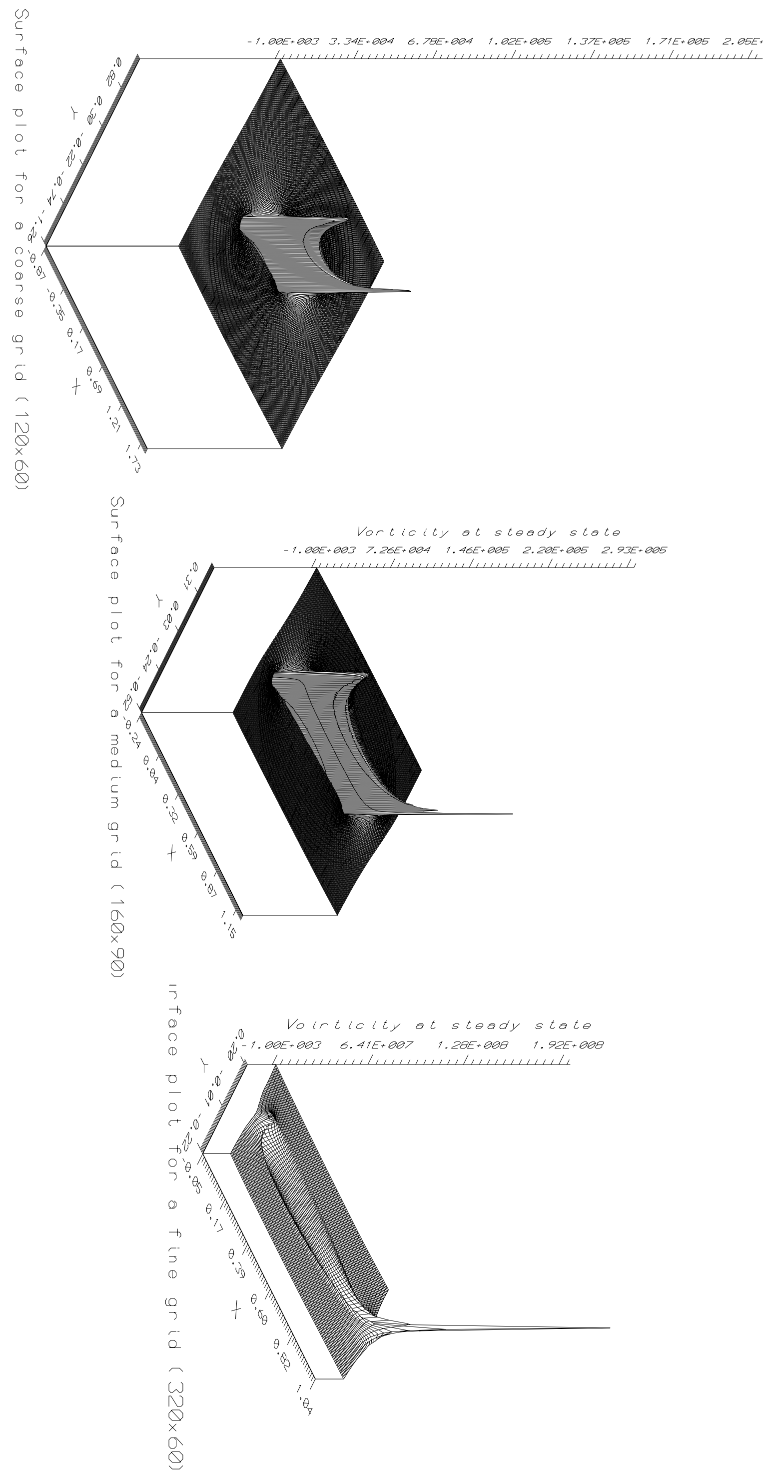
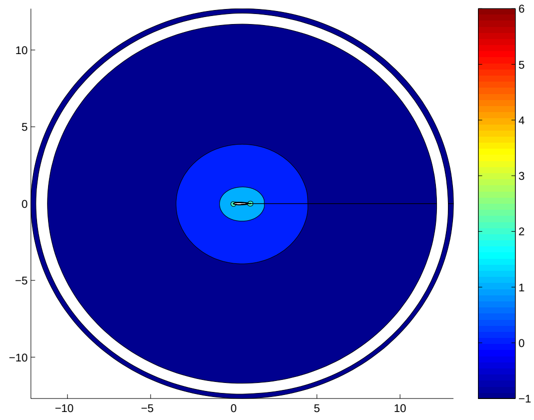
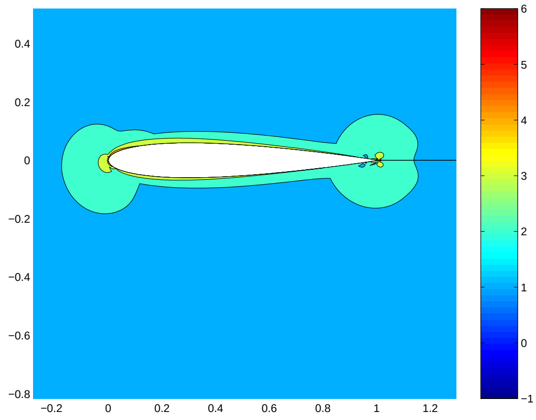


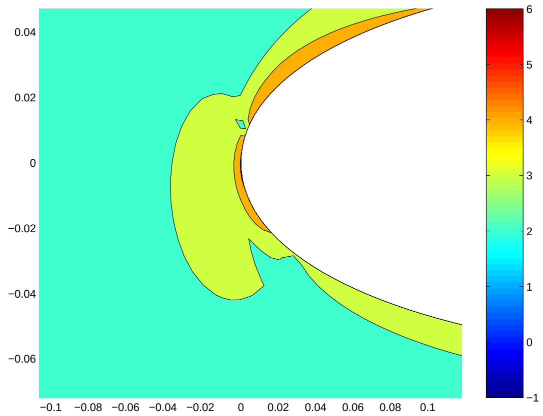
Figure 3: Surface Plots of Vorticity, at steady state, for different grids. ($M_\infty = 0.63, \alpha = 2^\circ$)



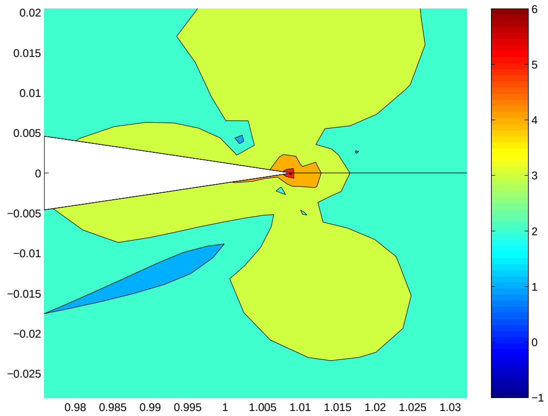
(a) The whole Computational Domain



(b) Close to the Airfoil

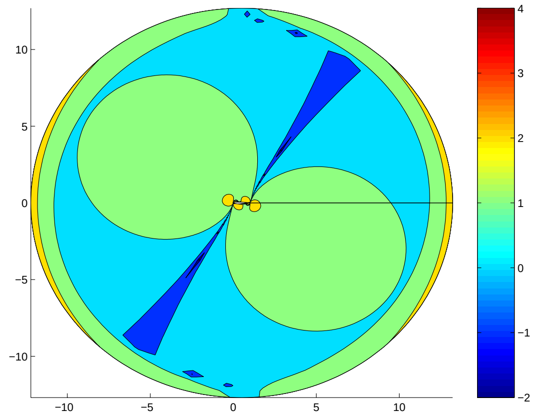


(c) The Leading edge

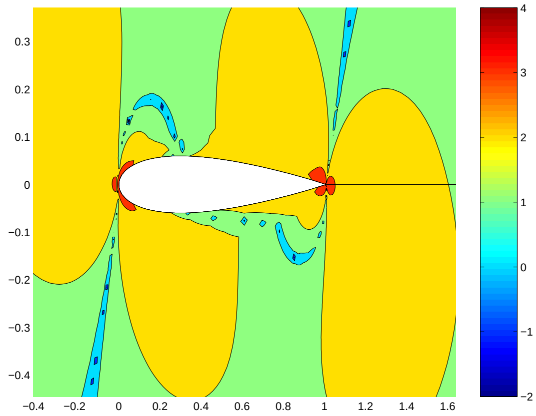


(d) The Trailing edge

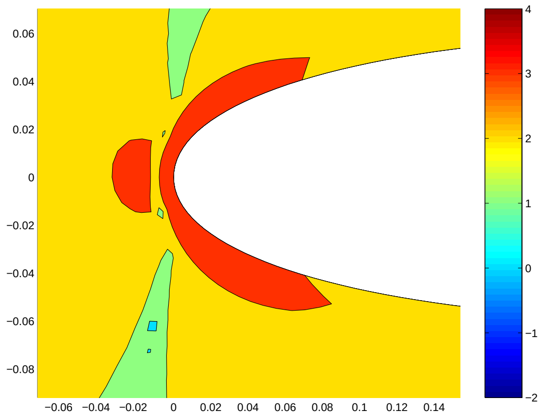
Figure 4: The Baroclinic term for the Flow past NACA 0012 at M_∞ = 0.16, α = 2.^{o} on Fine Grid



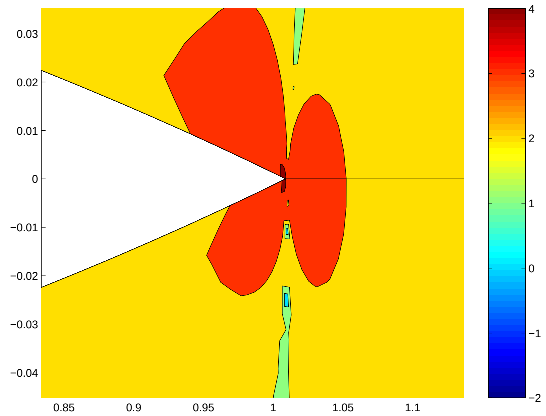
(a) The whole Computational Domain



(b) Close to the Airfoil



(c) The Leading edge



(d) The Trailing edge

Figure 5: Contours of Divergence for the Flow past NACA 0012 at $M_\infty = 0.16$, $\alpha = 2.^o$ on Fine Grid

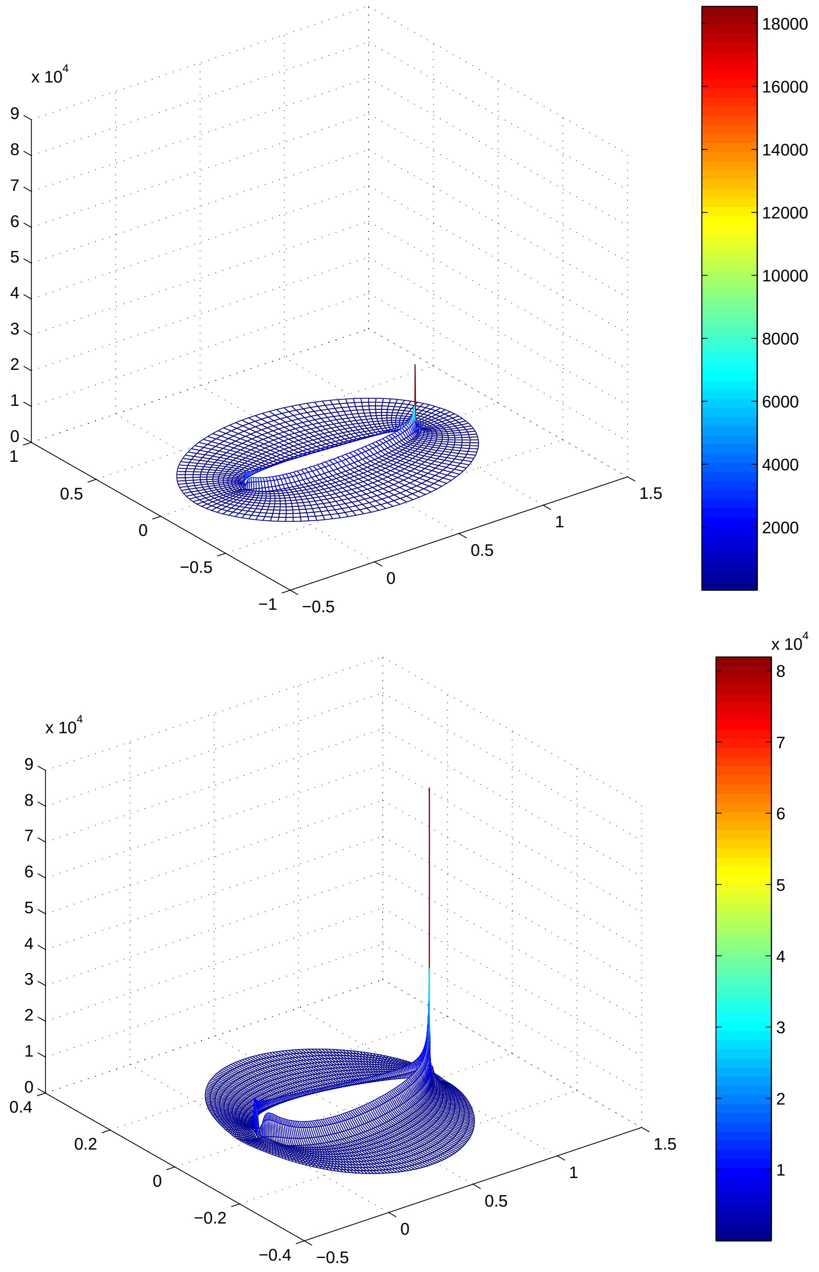


Figure 6: A 3-D Surface plot of the Vorticity distribution close to the airfoil for the coarse grid and the Fine grid

Conclusion

▷ An attempt to understand how circulation for lift is generated within the framework of inviscid compressible Flow.
▷ Baroclinic mechanism, activated by Flow tangency and the Kutta-Joukowski condition are responsible for generating vorticity and therefore circulation.
▷ Grid re£nement study :
the baroclinic term is a maximum at the trailing edge
Vorticity is very large at the trailing edge.