Extension of the Spectral Volume Method to the Convection/Diffusion Equation

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Abstract

The spectral volume (SV) method is a newly developed high-order finite volume method for hyperbolic conservation laws on unstructured grids. It has been successfully demonstrated for two-dimensional Euler equations. We wish to extend the SV method for solving the Navier-Stokes equations. As a first-step toward achieving that goal, the SV method is extended to the convention/diffusion equation,

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = \mu \frac{\partial^2 u}{\partial x^2} \ . \quad (1)$$

The key idea in the SV method is to sub-divide simplex cells (named spectral volumes) into sub cells (called control volumes), and the cells averages at the sub-cells are used to reconstruct a high-order polynomial in the macro-cell. Then the cell-averages at the sub-cells are updated using the usual finite volume approach. Let $C_{i,j}$ denote the j-th CV of i-th SV. If (1) is integrated in $C_{i,j}$, we obtain

$$\frac{d\overline{u}_{i,j}}{dt} + \frac{c}{h_{i,j}}\left\{ \hat{u}\big|_{i,j+\frac{1}{2}} - \hat{u}\big|_{i,j-\frac{1}{2}}) \right\} = \frac{\mu}{h_{i,j}}\left\{ \hat{u}_x\big|_{i,j+\frac{1}{2}} - \hat{u}_x\big|_{i,j-\frac{1}{2}}) \right\}, \quad (2)$$

where $\overline{u}_{i,j}$ is the cell-averaged variable at CV $C_{i,j}$, $h_{i,j}$ is the cell size, $\hat{u}\big|_{i,j+\frac{1}{2}}$ and $\hat{u}_x\big|_{i,j+\frac{1}{2}}$ are the "numerical inviscid and viscous fluxes" at the interfaces. At the macrocell boundaries, both the solution and derivative are discontinuous. The fluxes are not well defined. Obviously, the inviscid flux can be computed through "upwinding", i.e.,

$$\hat{u}\big|_{i,j+\frac{1}{2}} = \begin{cases} u\big|^{-}_{i,j+1/2} & if\ c > 0 \\ u\big|^{+}_{i,j+1/2} & otherwise \end{cases} . \quad (3)$$

For the viscous flux, the "common sense" suggests the simple averages

$$\hat{u}_x\big|_{i,j+\frac{1}{2}} = \left(u_x\big|^{-}_{i,j+1/2} + u_x\big|^{+}_{i,j+1/2} \right)/2 \ . \quad (4)$$

However this viscous flux produced a "wrong" numerical solution for the heat equation (c = 0) with the following initial condition $u(x,0) = \sin(x)$, as shown in Figure 1. To remedy this problem, a penalty term is added to the viscous flux, which takes the following form

$$\hat{u}_x\big|_{i,j+\frac{1}{2}} = \left(u_x\big|^{-}_{i,j+1/2} + u_x\big|^{+}_{i,j+1/2} \right)/2 + \varepsilon\left(u\big|^{+}_{i,j+1/2} - u\big|^{-}_{i,j+1/2} \right)/(h_{i,j} + h_{i,j+1}) \ . \quad (5)$$

A Fourier analysis is performed on the formulation, and it is determined that ε must be two to preserve the second-order accuracy for a linear reconstruction (a very nice integer!). The performance of the new flux is shown in Figure 2. In fact, the new viscous flux also works well for the convection/diffusion equation. Computational1results will be presented in the final paper.

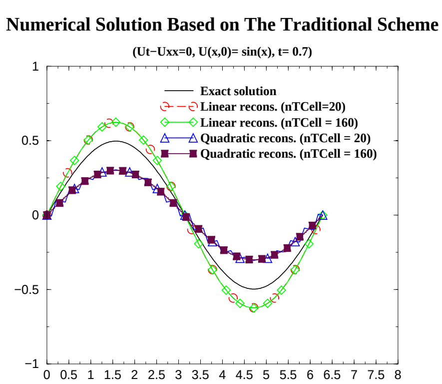

Figure 1.

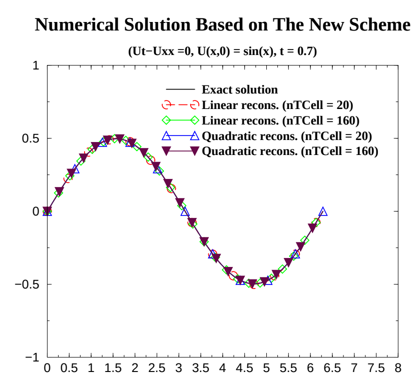

Figure 2.