Bayesian Sensitivity Analysis of a Simple Flutter Model

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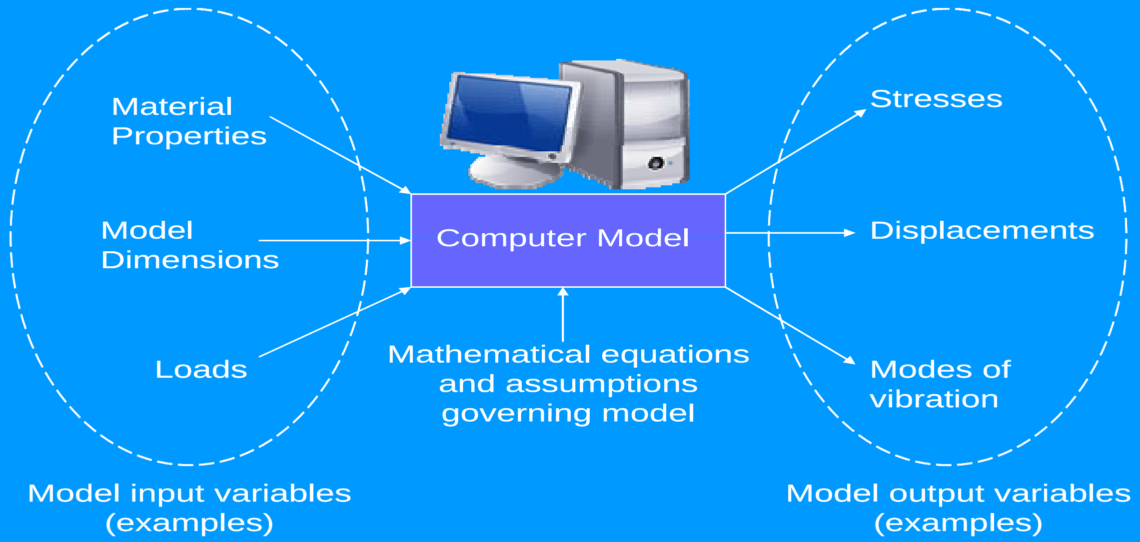
Overview

- Uncertainty and sensitivity analysis
- The Bayesian approach
- Simple Flutter
- Sensitivity analysis results
- Conclusions
- Questions and comments

Uncertainty in Modelling



Aleatoric

- Arising from inherent variability
- Machining tolerance, operating conditions
- Cannot be reduced

Epistemic

- "Model Imperfections"
- Simplifications, precise information unavailable
- Can be reduced

Sensitivity Analysis

- "How do individual model inputs contribute to the uncertainty in the output?"
- Why:
  - Increase robustness of model
  - Design optimisation
  - Identify parameters that require further research
  - Model simplification – eliminating variables
  - Greater understanding of model and variable interactions

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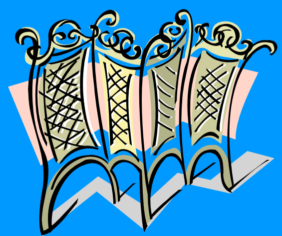
Sensitivity Analysis

Screening (qualitative ranking)	Local SA (Linear models, small perturbations)	Global SA (Often Monte Carlo)

Least Informative



Most Informative

Increasing computational cost →

Problems…

- Complex simulations can require a significant time for a single run
- Monte-Carlo techniques require many runs
- SA for several input variables can be unfeasible

A Solution – Bayesian Data Modelling

The Bayesian Approach

- ☞ Model treated as unknown function *f*(*x*)
- ☞ Input parameters represented as probability distributions (uniform or Gaussian for tractability)
- ☞ Gaussian process regression (GPR) used to build a metamodel from small number of model runs
- ☞ Sensitivity analysis data inferred directly from posterior distribution
- 👍 Application of GPR allows SA data to be collected for many fewer model runs, at comparable accuracy

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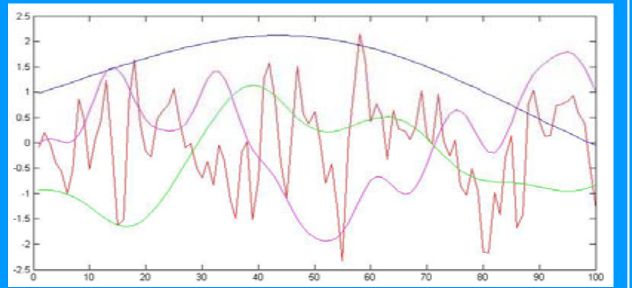
Gaussian Process Regression

Prior assumptions

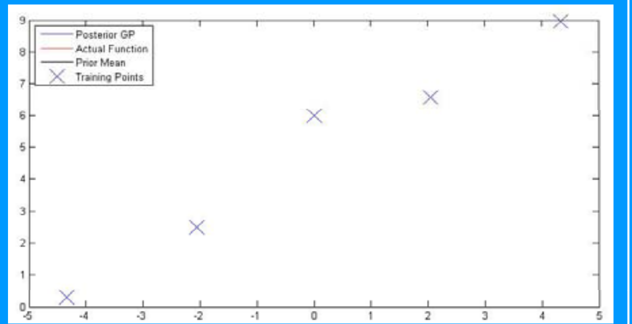
$$E\{f(\mathbf{x}) \mid \beta\} = \mathbf{h}(\mathbf{x})^T \beta$$

$$\text{cov}\{f(\mathbf{x}_i), f(\mathbf{x}_j) \mid \sigma^2, B\} = \sigma^2 \exp\{-(\mathbf{x}_i - \mathbf{x}_j)^T B(\mathbf{x}_i - \mathbf{x}_j)\}$$



Training data

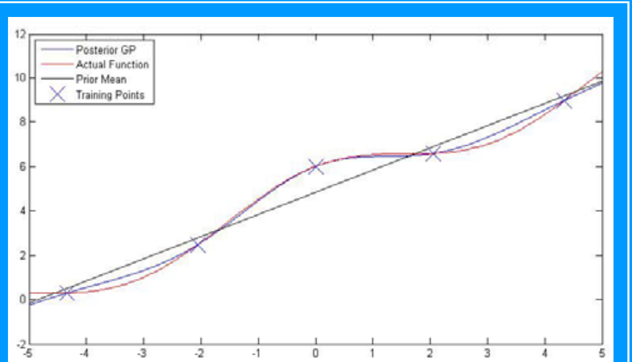
$$\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_n\}$$
$$\mathbf{y} = \{y_1, y_2, ..., y_n\}$$



Hyperparameter estimation, condition on training data

Posterior distribution

$$[f(\mathbf{x}) \mid B, \mathbf{y}] \sim t_{n-q}\{m^*(\mathbf{x}), \hat{\sigma}^2 c^*(\mathbf{x}, \mathbf{x}')\}$$



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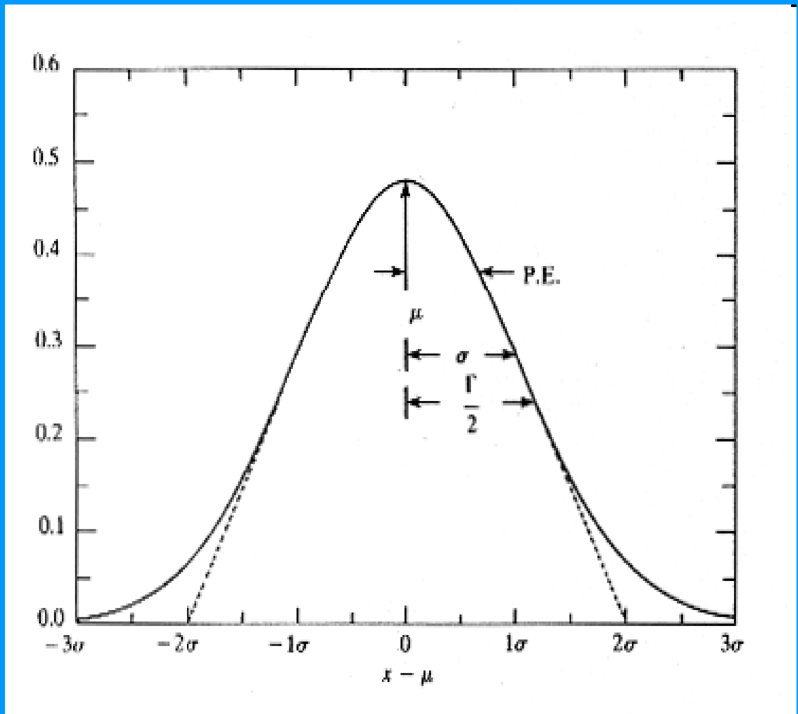

Uncertainty in output

- Mean $E^*\{E(Y)\}$
- Variance

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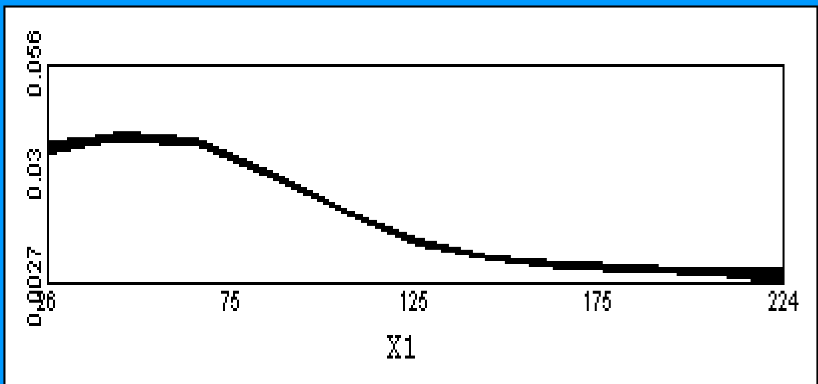
Posterior distribution

$$[f(\mathbf{x}) \mid B, \mathbf{y}] \sim t_{n-q}\{m^*(\mathbf{x}), \hat{\sigma}^2 c^*(\mathbf{x}, \mathbf{x}')\}$$

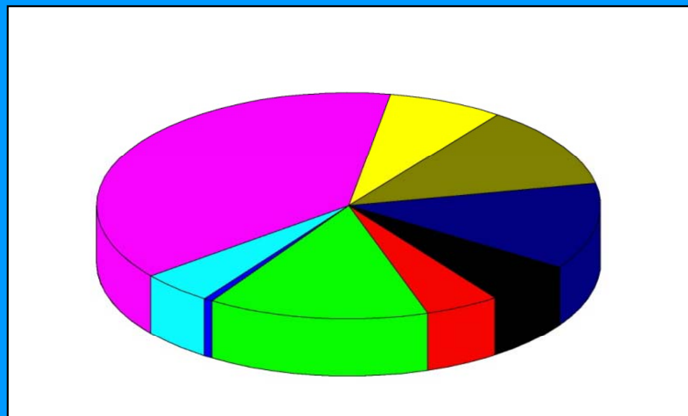
Main effects & interactions

$$E^*\{E(Y \mid \mathbf{x}_p)\} = \int_{\chi_{-p}} m^*(\mathbf{x}) dG_{-p|p}(\mathbf{x}_{-p} \mid \mathbf{x}_p)$$

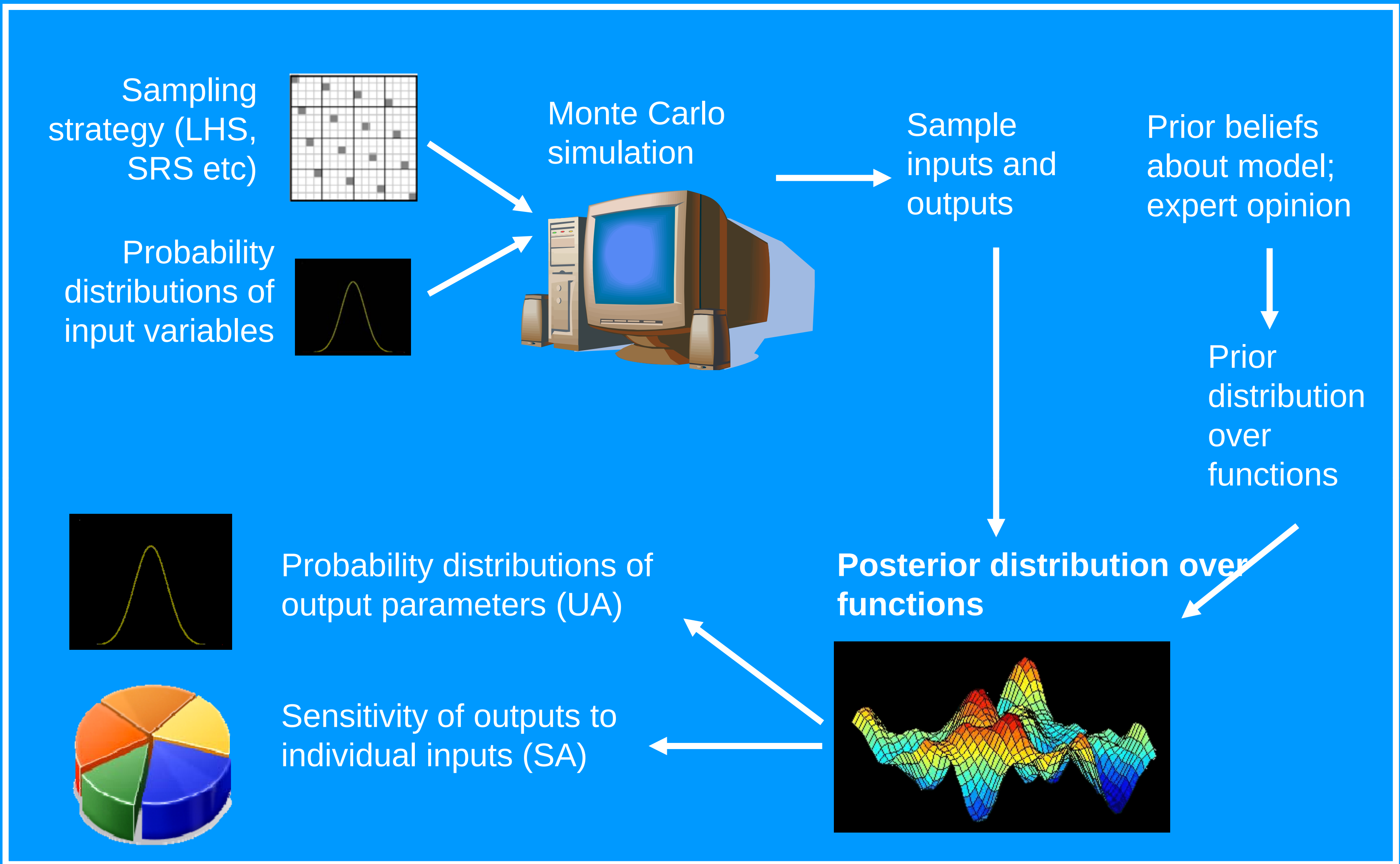


Sensitivity indices

$$E^*[\mathrm{var}\{E(Y \mid X_p)\}]$$
$$= E^*[E\{E(Y \mid X_p)^2\}] - E^*\{E(Y)^2\}$$



A Bayesian Approach



Binary Flutter

Basic Equation is (notation: Wright and Cooper).

$$[A]\{\ddot{z}\}+(\rho V[B]+[D])\{\dot{z}\}+(\rho V^2[C]+[E])\{z\}=0$$

[*A*], [*B*], [*C*], [*D*] and [*E*] represent: structural inertia, aerodynamic damping, aerodynamic stiffness, structural damping and structural stiffness.

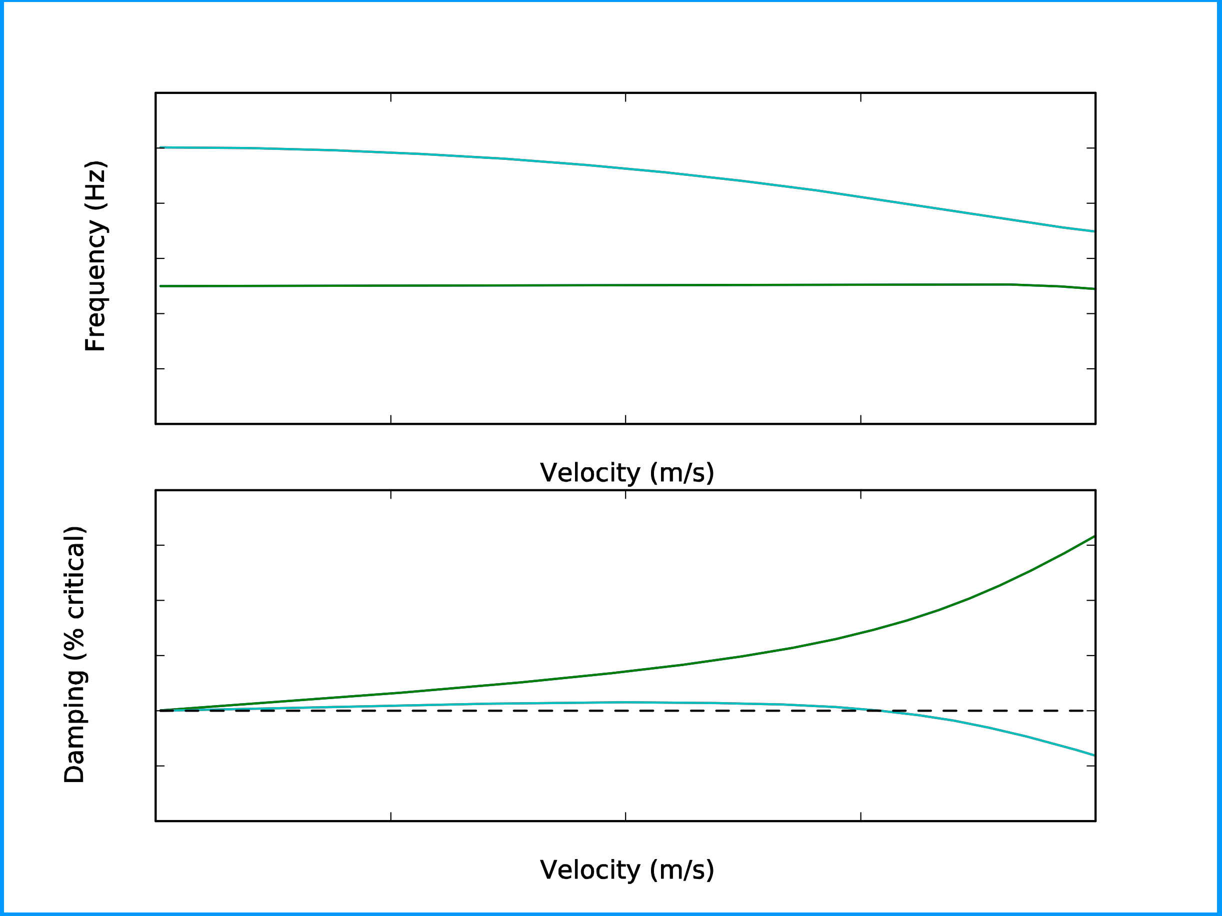
{*z*} is a 2-vector representing flap and pitch degrees of freedom for a rigid rectangular wing.

Baseline Parameters

Certain		***Uncertain***	
Semi-span *s*	**7.5 m**	**Flap stiffness**	$I_{\gamma}(5\times 2\pi)^2$ **Nm/rad**
Chord *c*	**2.0 m**	**Pitch stiffness**	$I_{\theta}(10\times 2\pi)^2$ **Nm/rad**
Flexural axis *xf*	**0.48 *c***	**Lift curve slope *aw***	***2π***
Mass axis *xm*	**0.5 *c***	**Nondimensional pitch Damping derivative *Mthetadot***	***-1.2***
Mass per unit area	**100 kg/m^2**	**Air density *rho***	***1.225 kg/m^2***

The uncertain parameters are allowed to vary by 10% around the nominal values.

Baseline Results



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The Sensitivity Analysis

- Maximin latin hypercube sampling
- 200 model runs
- Squared-exponential covariance function (assumes smooth response)
- Inputs assumed uncorrelated
- Gem-SA used for DOE and analysis

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Main Effects

Variable	Main Effect
Kf	5.03
Kt	78.65
A	3.03
Mdt	2.79
Rho	10.36
Total	99.9953

No significant interactions

Predictive posterior mean = 154.147

Predictive posterior SD = 13.458

Comparison

GEM-SA
Predictive posterior mean = 154.147
Predictive posterior SD = 13.458

MC (200 runs)
Predictive posterior mean = 153.67
Predictive posterior SD = 13.443

MC (20000 runs)
Predictive posterior mean = 154.108
Predictive posterior SD = 13.455

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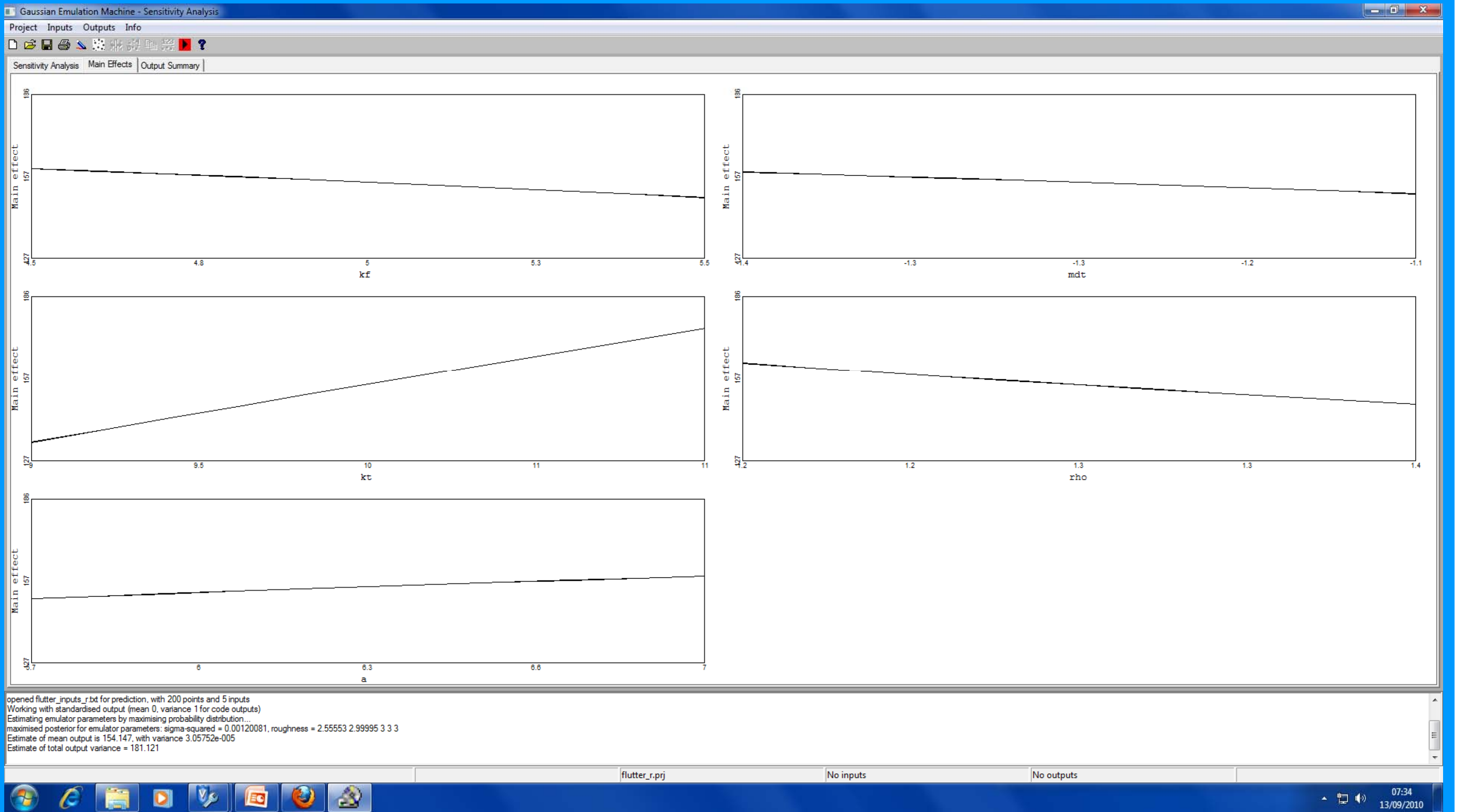

Conclusions

- Bayesian sensitivity analysis allows detailed insight into large, nonlinear uncertain models.
- The model here is trivial; however, a real flutter model would couple in a structural FE model and the benefits would be felt.
- Assumptions used (smoothness of model, input distributions etc), thus uncertainty results uncertain! However, good indicator.