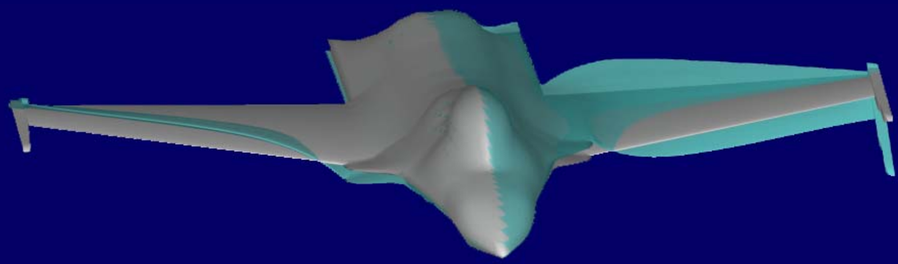
E C E R T A – Enabling Certification by Analysis



Eigenvalue Stability Formulation for Uncertainty Propagation to Nonlinear Aeroelastic Systems

S. Marques

University of Liverpool, June 2010

UNIVERSITY OF LIVERPOOL

Marie Curie
Excellence Team

LABORATORY

www.cfd4aircraft.com

CONTENTS

- Motivation
- Schur Complement Method
- Uncertainty quantification
- Routes for variability propagation
- Conclusions

MOTIVATION

- Aeroelastic flight test variability

MOTIVATION

- Aeroelastic flight test variability
- Model variability

MOTIVATION

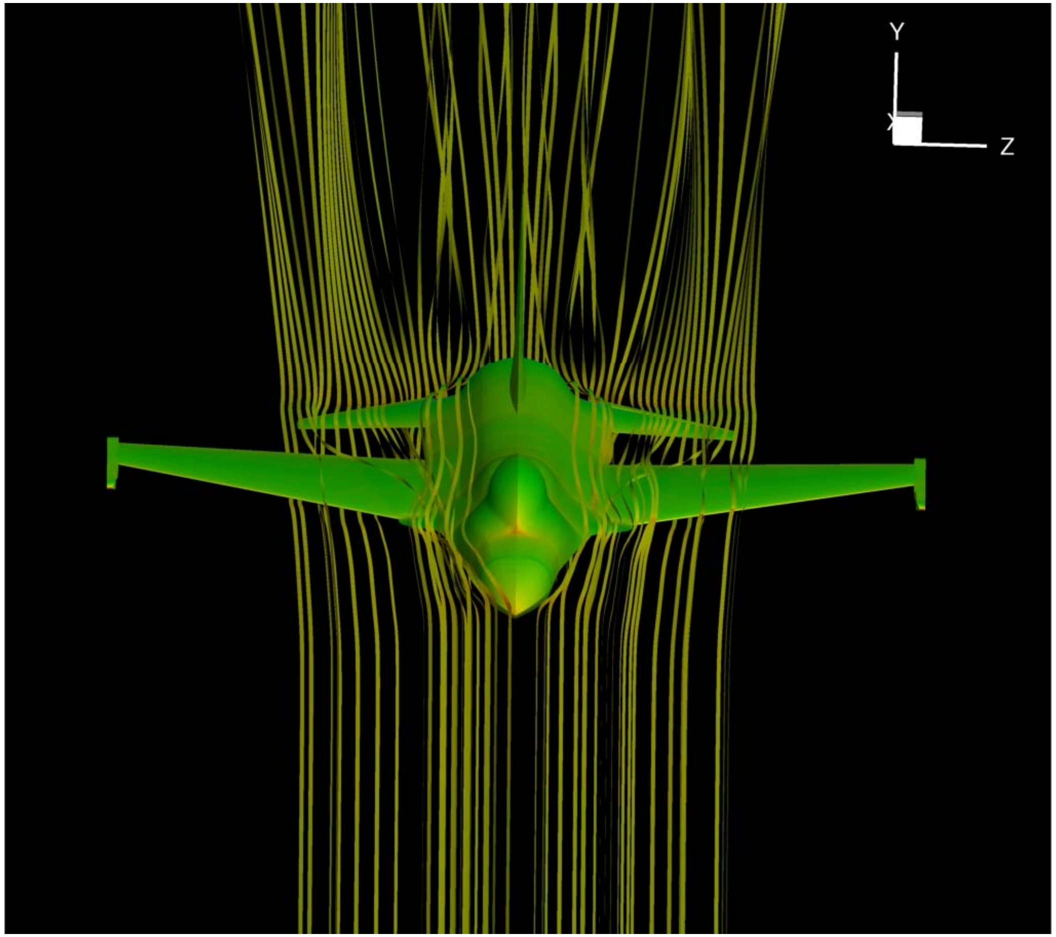
- Aeroelastic flight test variability
- Model variability
- Variability is present and has an impact in aeroelasticity

AIMS

- Study aeroelasticity stability throughout the flight envelope

AIMS

• Study aeroelasticity stability throughout the flight envelope
  – CFD



AIMS

- Study aeroelasticity stability throughout the flight envelope
  - CFD

- Analyse the influence of structural variability

AIMS

- Study aeroelasticity stability throughout the flight envelope
  - CFD

- Analyse the influence of structural variability
  - Non-deterministic approach

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- Search parameter space

AIMS

- Study aeroelasticity stability throughout the flight envelope
  - CFD
- Analyse the influence of structural variability
  - Non-deterministic approach
- Search parameter space
  - Fast method

CFD BASED EIGENVALUE SOLVER

SCHUR METHOD

SCHUR METHOD

$$\begin{bmatrix} A_{ff} & A_{fs} \\ A_{sf} & A_{ss} \end{bmatrix} \begin{bmatrix} p_f \\ p_s \end{bmatrix} = \lambda \begin{bmatrix} p_f \\ p_s \end{bmatrix}$$

- Schur Complement formulation:

$$\mathbf{E} = S(\lambda)\mathbf{p}_s - \lambda \mathbf{p}_s = 0$$

$$S(\lambda) = A_{ss} - A_{sf}(A_{ff} - \lambda I)^{-1} A_{fs}$$

Bekas and Saad, SIAM Journal of Scientific Computing 27(2) 458, 2005
Badcock and Woodgate, AIAA Journal 48(6), 2010

SCHUR METHOD

- The new formulation is solved by Newton´s Method

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

Small order nonlinear eigenvalue problem

$$\mathbf{u} = [\mathbf{p} \ \boldsymbol{\lambda}]^T$$

$$\mathbf{E}(\mathbf{w}_0, \lambda, \mathbf{p}_s, \phi, \omega) = \left(A_{ss} - A_{sf}(A_{ff} - \lambda I)^{-1} A_{fs}\right)\mathbf{p}_s - \lambda \mathbf{p}_s$$

SCHUR METHOD

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E} = S(\lambda)\mathbf{p}_s - \lambda \mathbf{p}_s$$

SCHUR METHOD

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

Full residual evaluation on RHS

Series approximation used for Jacobian on LHS

$$\mathbf{E} = S(\lambda)\mathbf{p}_s - \lambda \mathbf{p}_s$$

$$(A_{ff} - \lambda I)^{-1} \approx A_{ff}^{-1} + \lambda A_{ff}^{-1} A_{ff}^{-1} + \lambda^2 A_{ff}^{-1} A_{ff}^{-1} A_{ff}^{-1} + \ldots$$

SCHUR METHOD FOR UQ

- Monte-Carlo Analysis

SCHUR METHOD FOR UQ

- Monte-Carlo Analysis
- Interval Analysis

MONTE-CARLO ANALYSIS

- Generate Random Parameters
  - Type of distribution (normal, uniform, etc)
  - Variation Amplitude (σ)
  - Generate random inputs (structural modes)

MONTE-CARLO ANALYSIS

- Generate Random Parameters
  - –Calculate aeroelastic eigenvalues
  - – Contruct PDF for eigenvalues and flutter altitude

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E} = S(\lambda)\mathbf{p}_s - \lambda \mathbf{p}_s$$

Mean Parameter series approximation for LHS

Full residual evaluation for each randomised mode set

INTERVAL ANALYSIS

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•Assume chosen parameters are in a chosen interval:

$$\underline{\theta_i} < \theta_i < \overline{\theta_i}$$

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• Optimisation problem solved using fmincon (matlab)

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- Optimisation problem solved using fmincon (matlab)
  - Generates new modes from MSC. Nastran
  - Calls Schur calculation multiple times

INTERVAL ANALYSIS

- Optimisation problem solved using fmincon (matlab)
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$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E} = S(\lambda)\mathbf{p}_s - \lambda \mathbf{p}_s$$

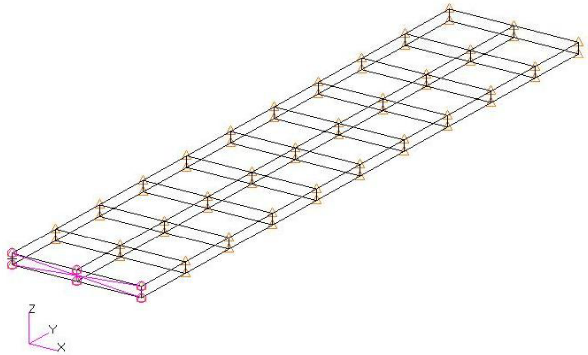
Mid-Interval Parameter series approximation for LHS

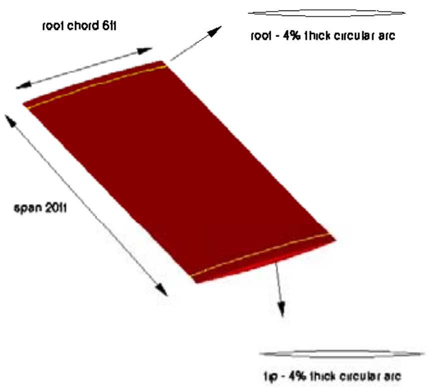
Full residual evaluation for each parameter set mode

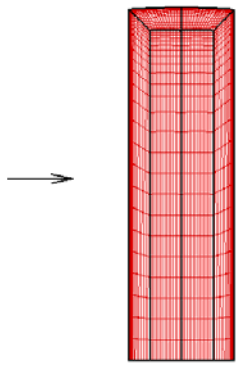
STRUCTURAL VARIABILTIY

GOLAND WING

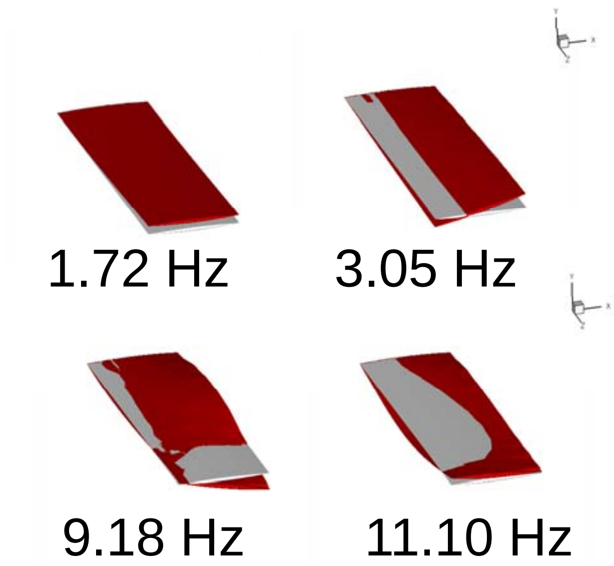
GOLAND WING

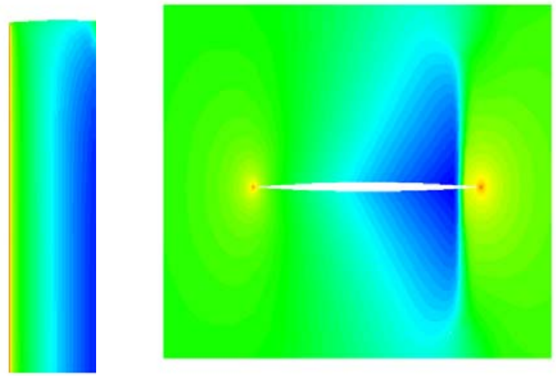






236k points



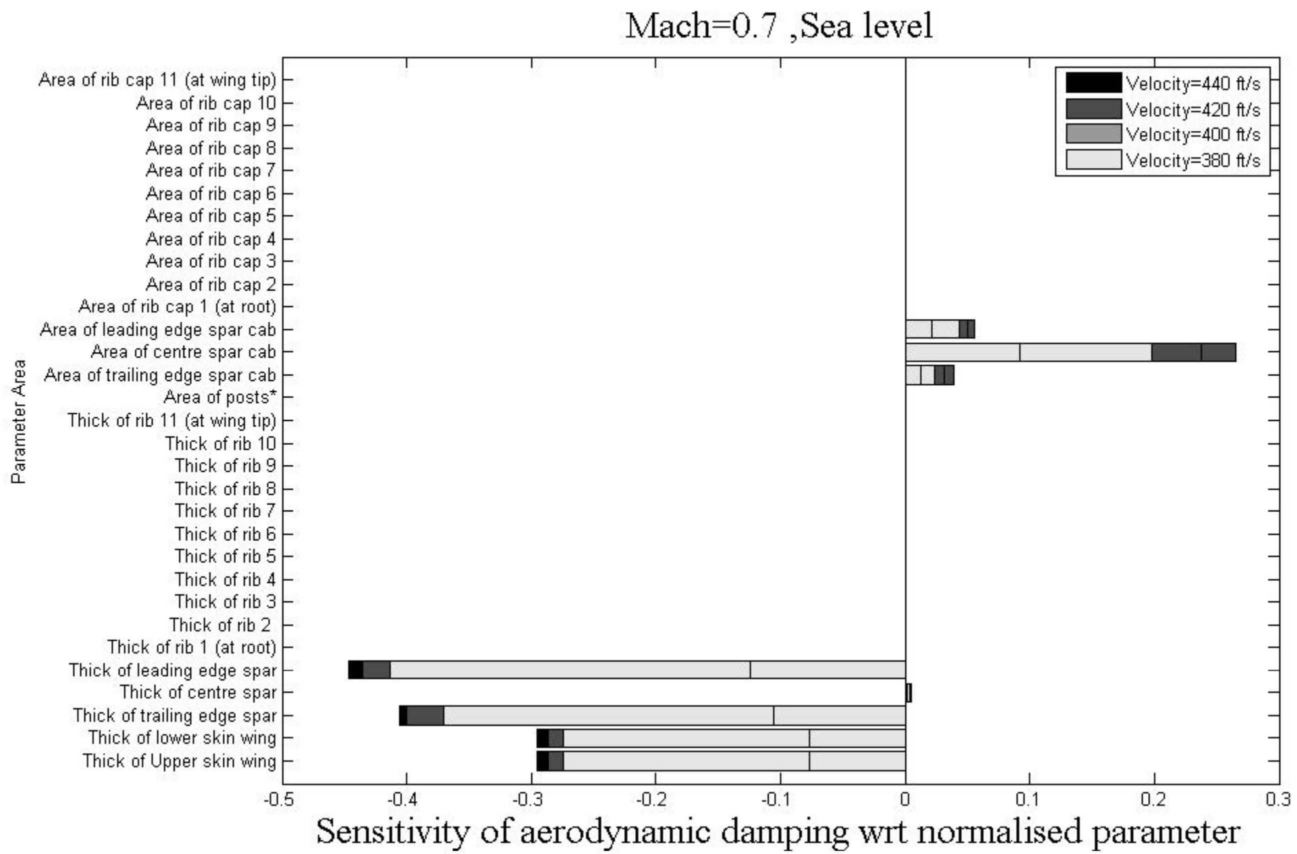


Mach 0.92

MONTE-CARLO ANALYSIS

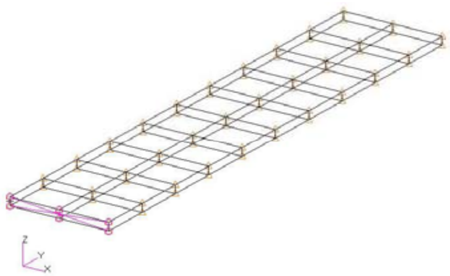
- 7 Structural Parameters

STRUCTURAL COMPONENTS

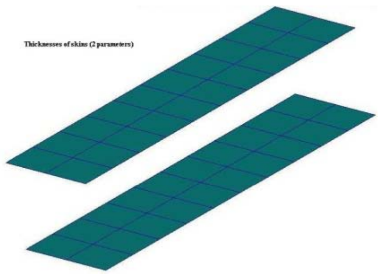


7 Structural parameters significantly influence flutter – vector θ

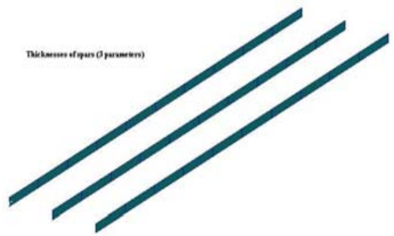
STRUCTURAL COMPONENTS



Thicknesses of skins

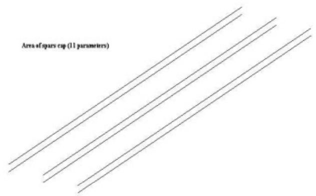


Thicknesses of spars

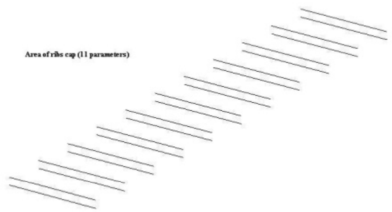


Thicknesses of ribs

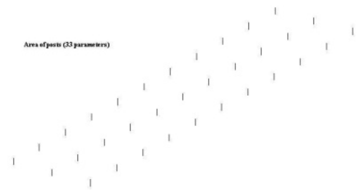
Area of spars cap



Area of ribs cap



Area of posts



MONTE-CARLO ANALYSIS

- 7 Structural Parameters

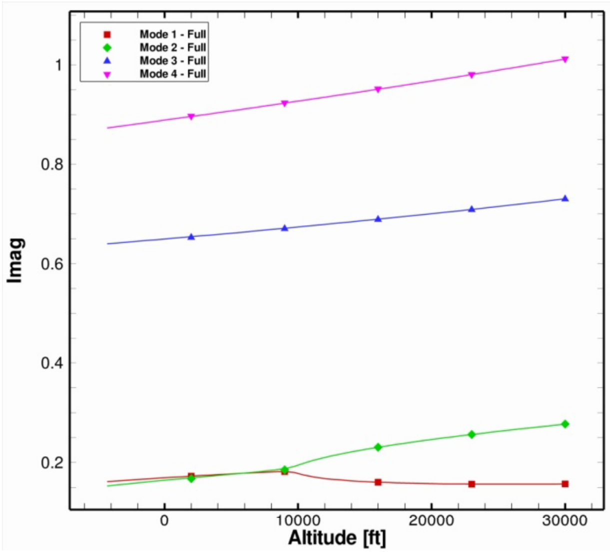
  – Variation within ±5%, having a normal distribution

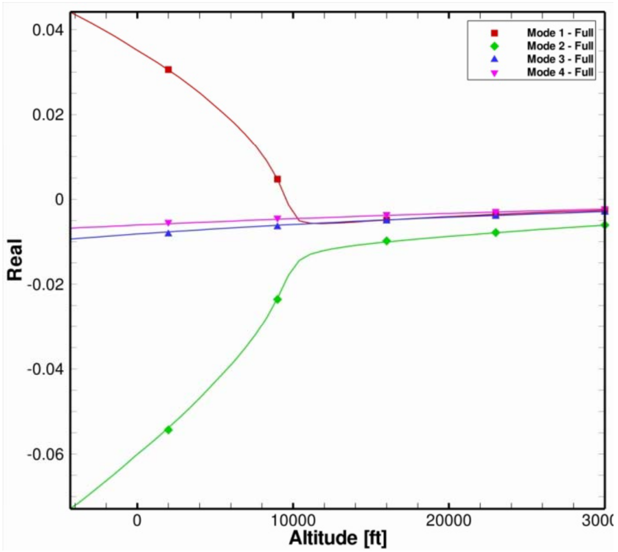
  – Generate 1000 structural mode sets and frequencies (MSC. Nastran)

  – Calculate aeroelastic eigenvalues for 1000 sets

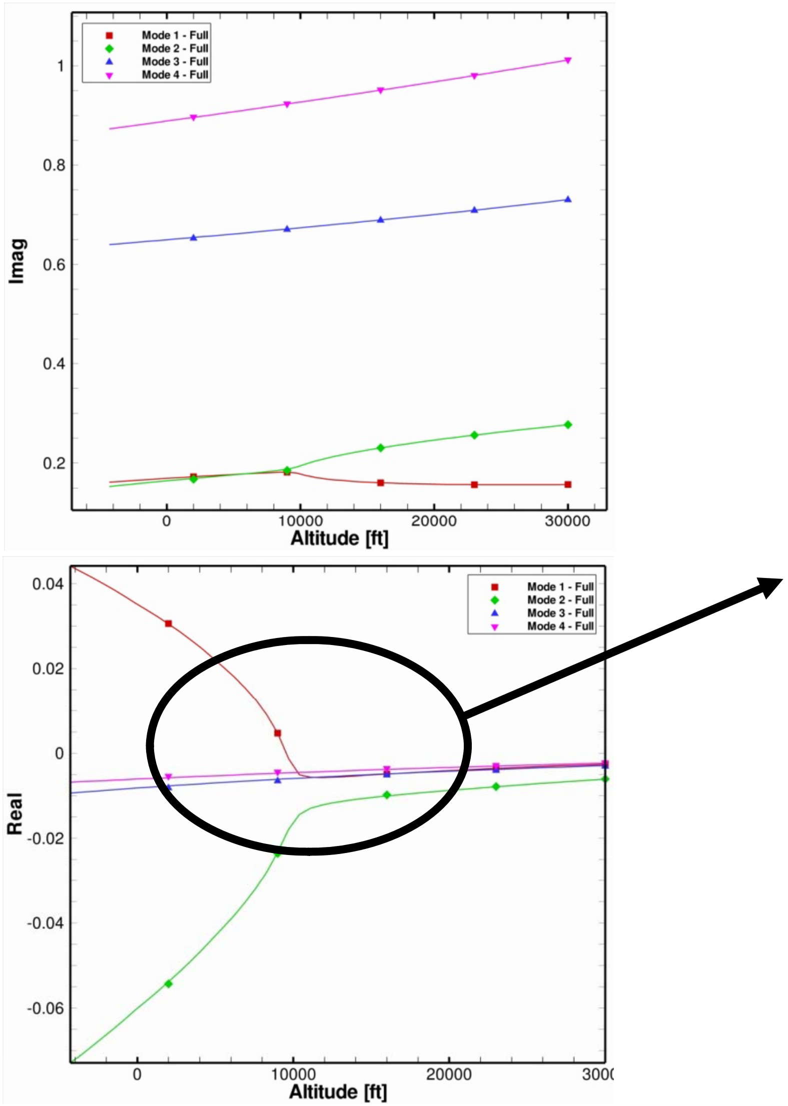
  – Contruct PDF for eigenvalues and flutter altitude

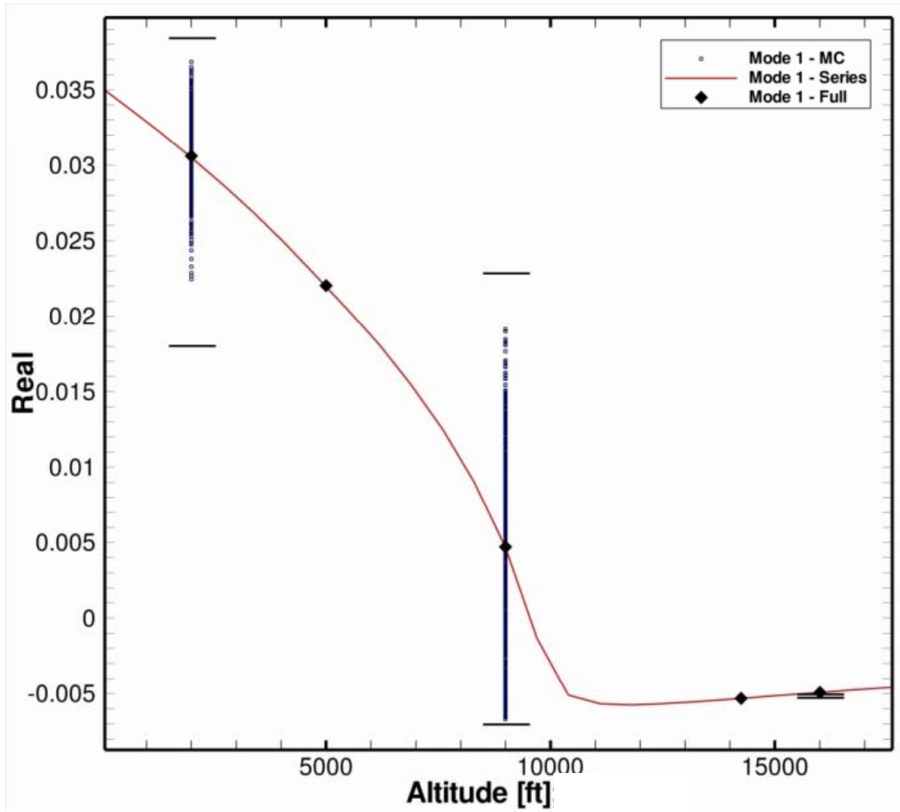
GOLAND WING CLEAN – M0.50





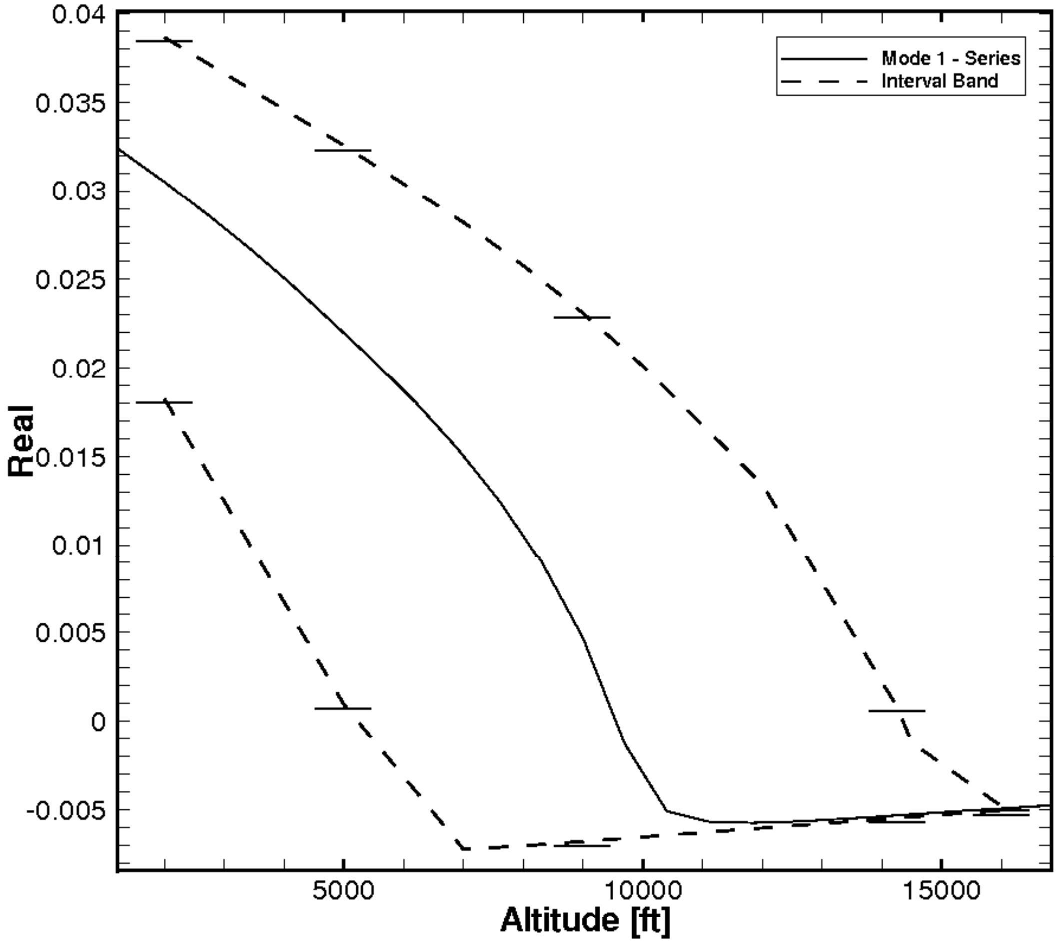
GOLAND WING CLEAN – M0.50



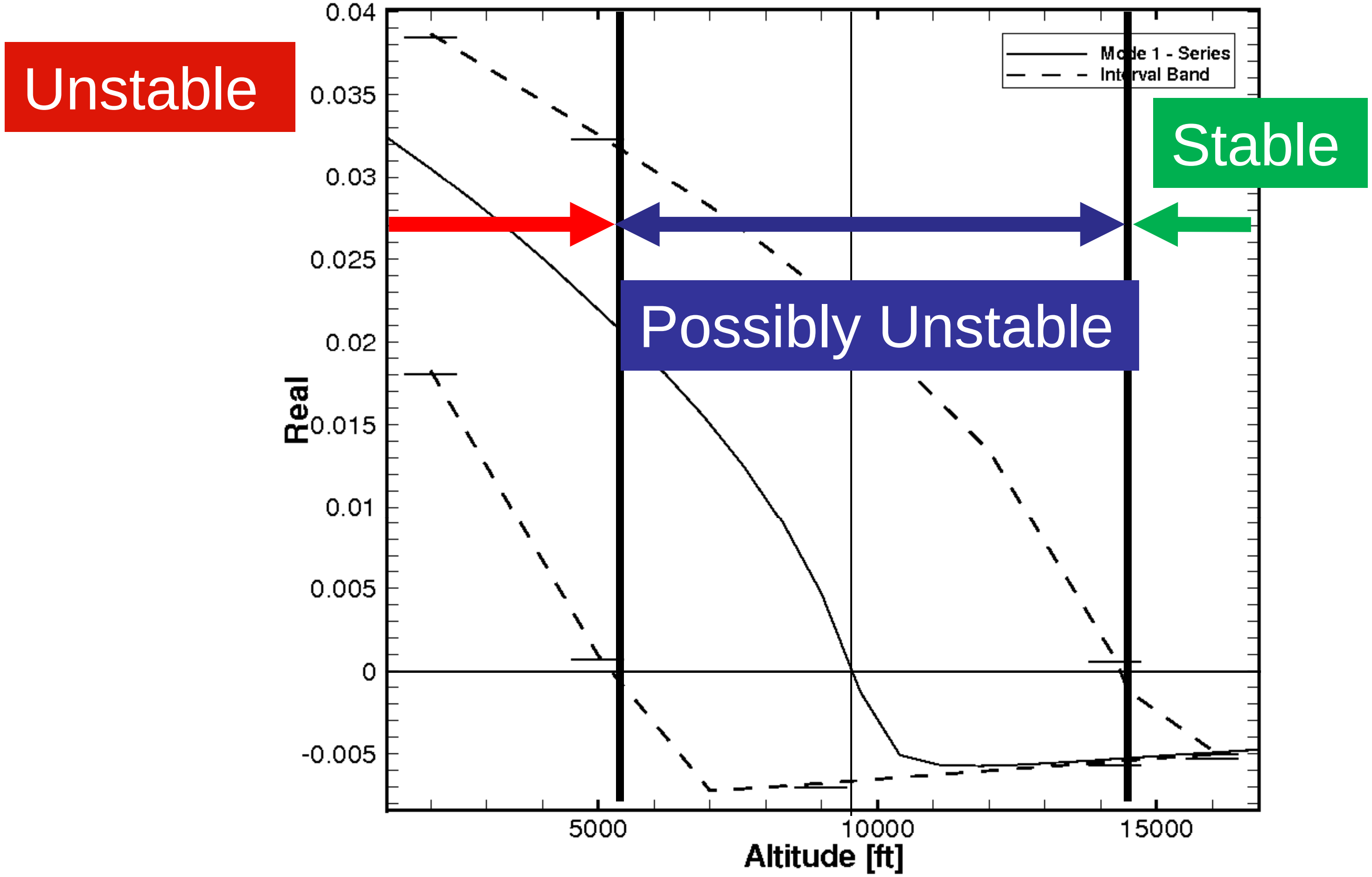


5% Variation of parameters

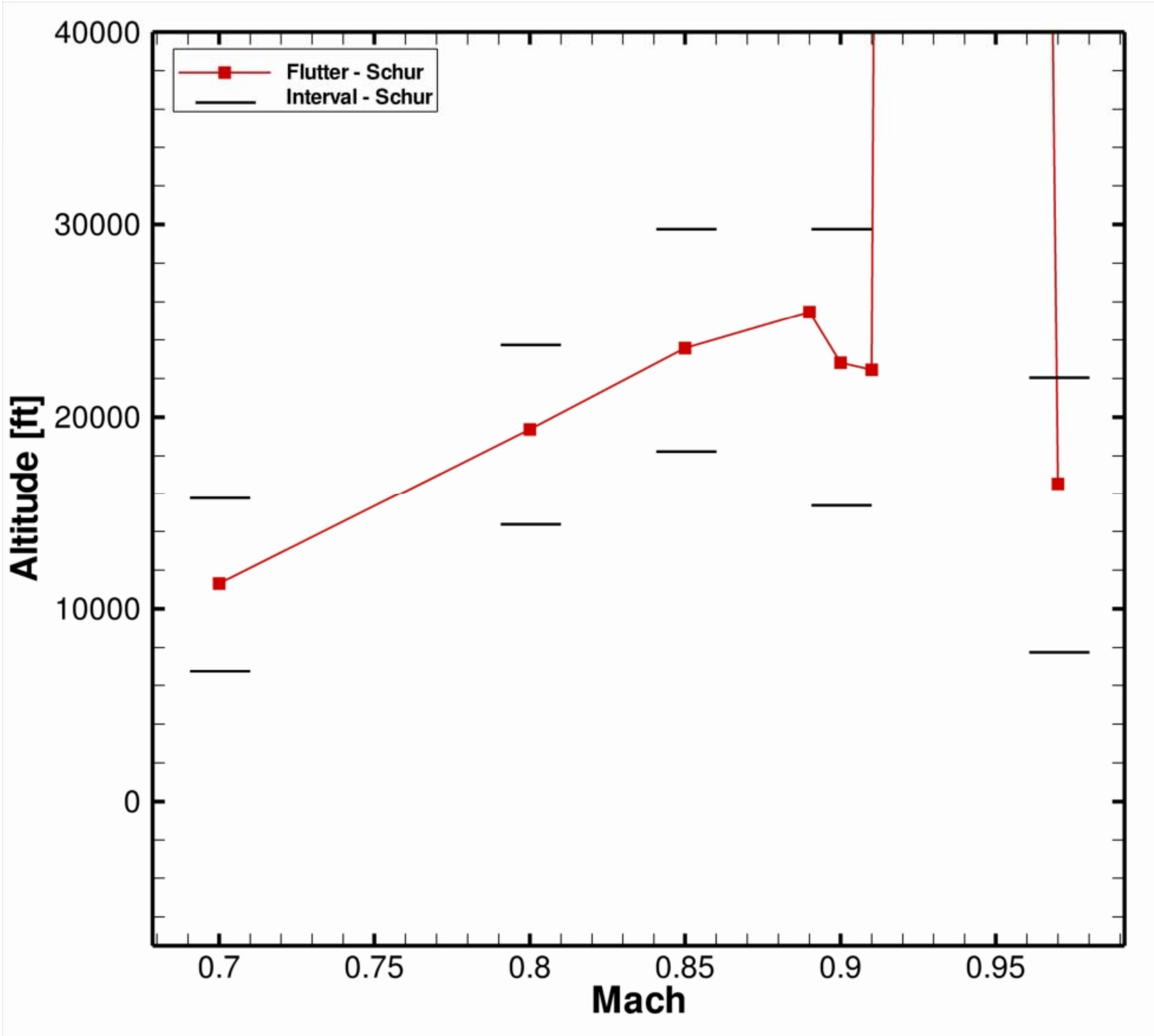
STABILITY BOUNDARY



STABILITY BOUNDARY

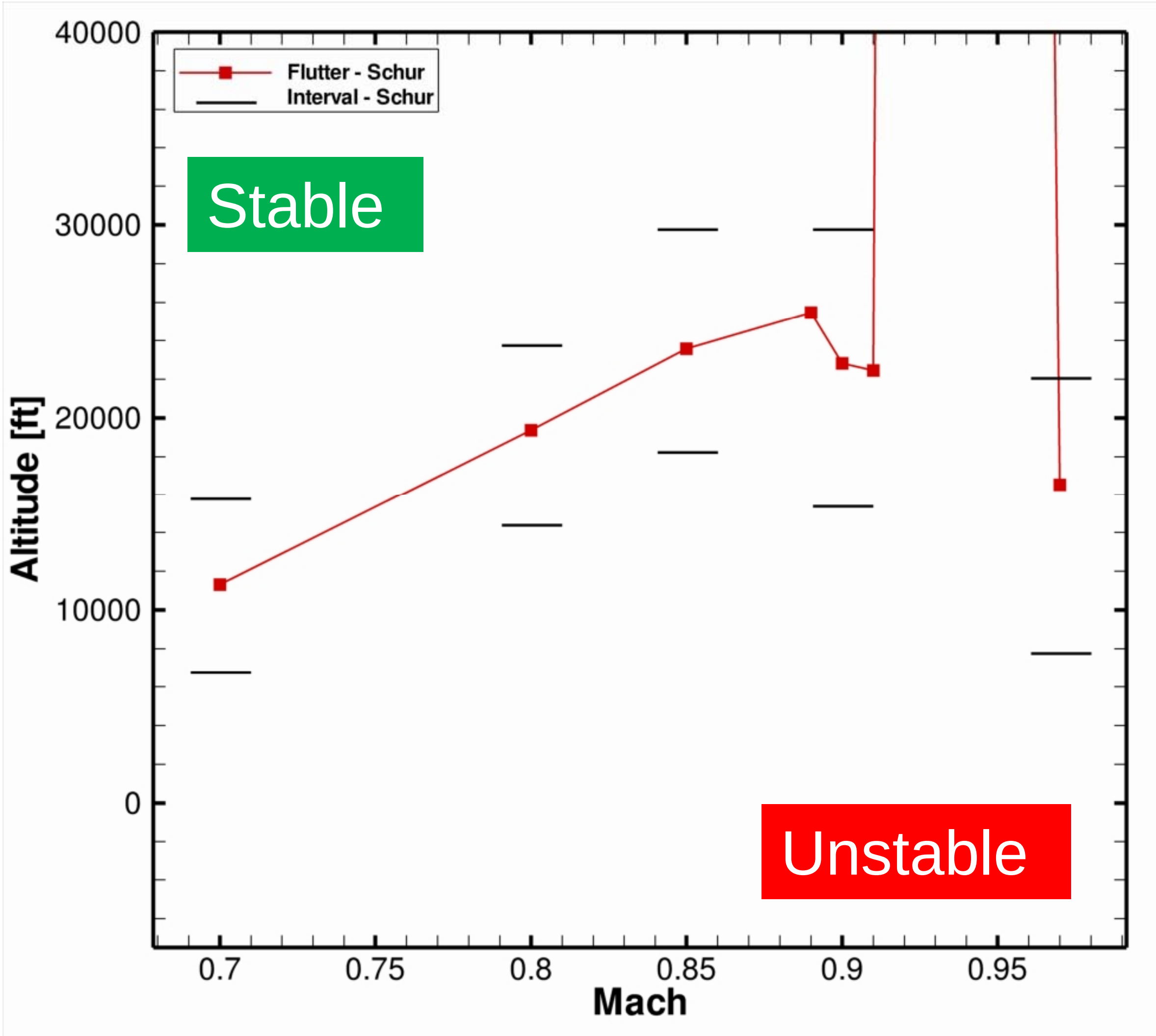


GOLAND WING^{+}



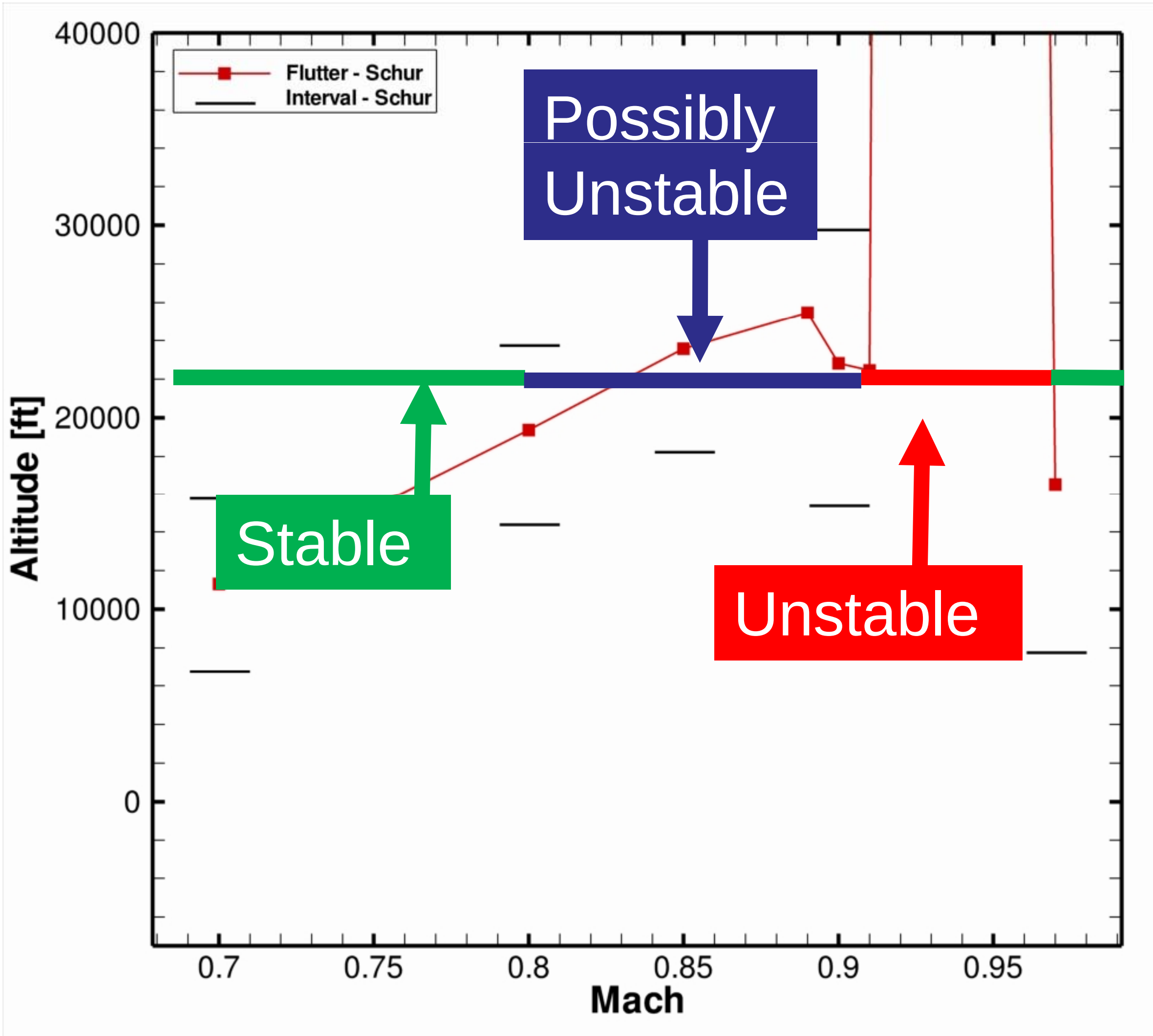
Flutter - Schur
Interval - Schur
40000
30000
20000
10000
0
Altitude [ft]
0.7
0.75
0.8
0.85
0.9
0.95
Mach

GOLAND WING+

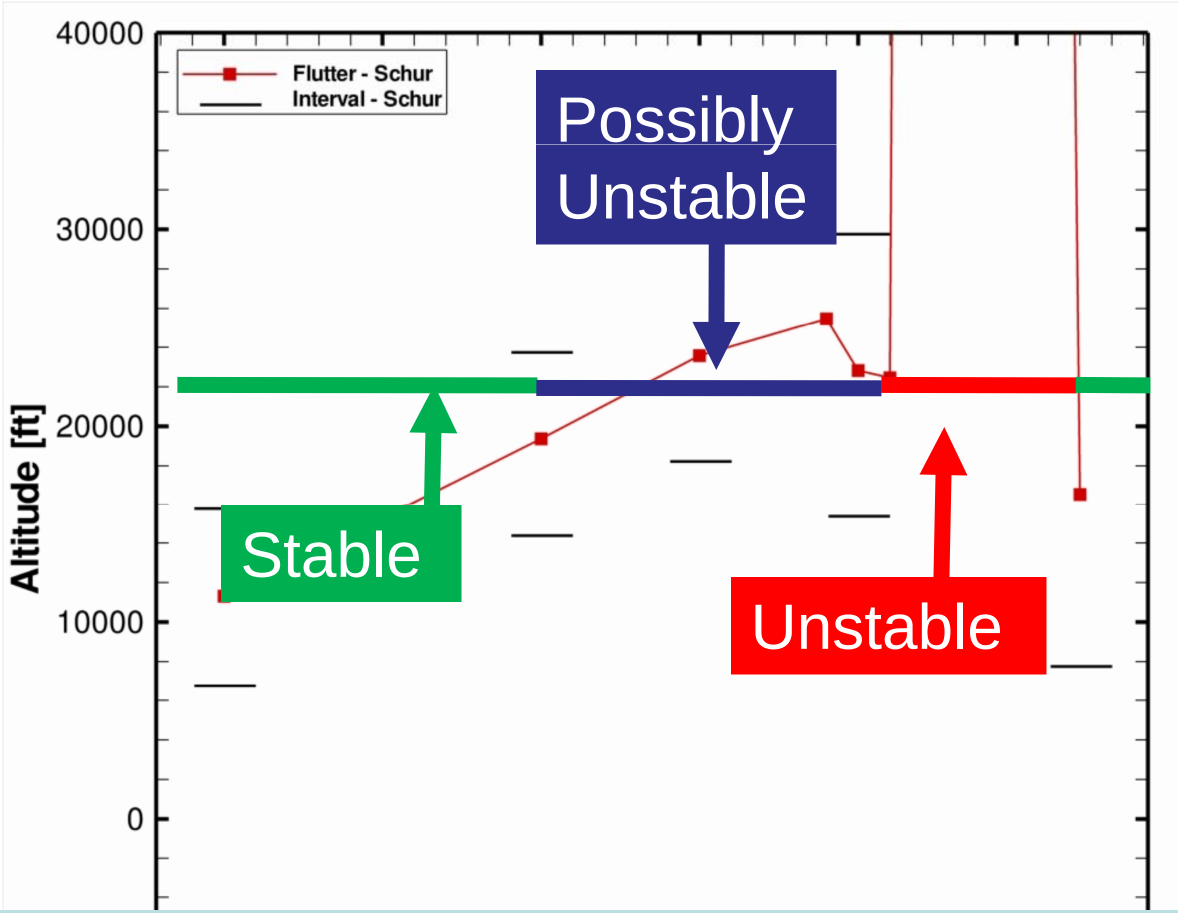


Flutter - Schur
Interval - Schur
Stable
Unstable
Altitude [ft]
40000
30000
20000
10000
0
0.7
0.75
0.8
0.85
0.9
0.95
Mach

GOLAND WING+


Flutter - Schur
Interval - Schur
Possibly Unstable
Stable
Unstable
40000
30000
20000
10000
0
Altitude [ft]
0.7
0.75
0.8
0.85
0.9
0.95
Mach

GOLAND WING+



Marques, S., Badcock, K.J., Khodaparast, H.H. and Mottershead, J.E., Transonic Aeroelastic Stability Predictions Under the Influence of Structural Variability, **Journal of Aircraft**, 47(4), 2010, 1229-1239

ROUTES FOR STRUCTURAL UNCERTAINTY

SCHUR METHOD

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

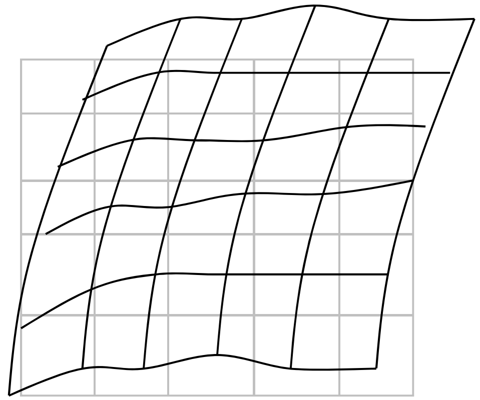
$$\mathbf{E}(\mathbf{w}_0, \lambda, \mathbf{p}_s, \phi, \omega) = \left[(A_{ss} - \lambda I) - A_{sf} (A_{ff} - \lambda I)^{-1} A_{fs} \right] \mathbf{p}_\mathbf{s} - \lambda \mathbf{p}_\mathbf{s}$$

$$R_s = \varphi_i^T S^T f_{surf} - \omega_i^2 \alpha_i$$

SCHUR METHOD

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E}(\mathbf{w}_0, \lambda, \mathbf{p}_s, \phi, \omega) = \left[(A_{ss} - \lambda I) - A_{sf} (A_{ff} - \lambda I)^{-1} A_{fs} \right] \mathbf{p}_s - \lambda \mathbf{p}_s$$



SCHUR METHOD

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E}\left(\mathbf{w}_0, \lambda, \mathbf{p}_s, \phi, \omega\right) = \left[\left(A_{ss} - \lambda I\right) - A_{sf}\left(A_{ff} - \lambda I\right)^{-1} A_{fs}\right]\mathbf{p}_\mathbf{s} - \lambda \mathbf{p}_\mathbf{s}$$

MOTIVATION

• Influence of structural variability on aeroelastic response:

→ Natural frequencies

MOTIVATION

• Influence of structural variability on aeroelastic response:

→ Natural frequencies | Route 1

MOTIVATION

• Influence of structural variability on aeroelastic response:

- Natural frequencies — Route 1
- Mode Shapes

MOTIVATION

• Influence of structural variability on aeroelastic response:

- Natural frequencies — Route 1
- Mode Shapes+R1 — Route 2

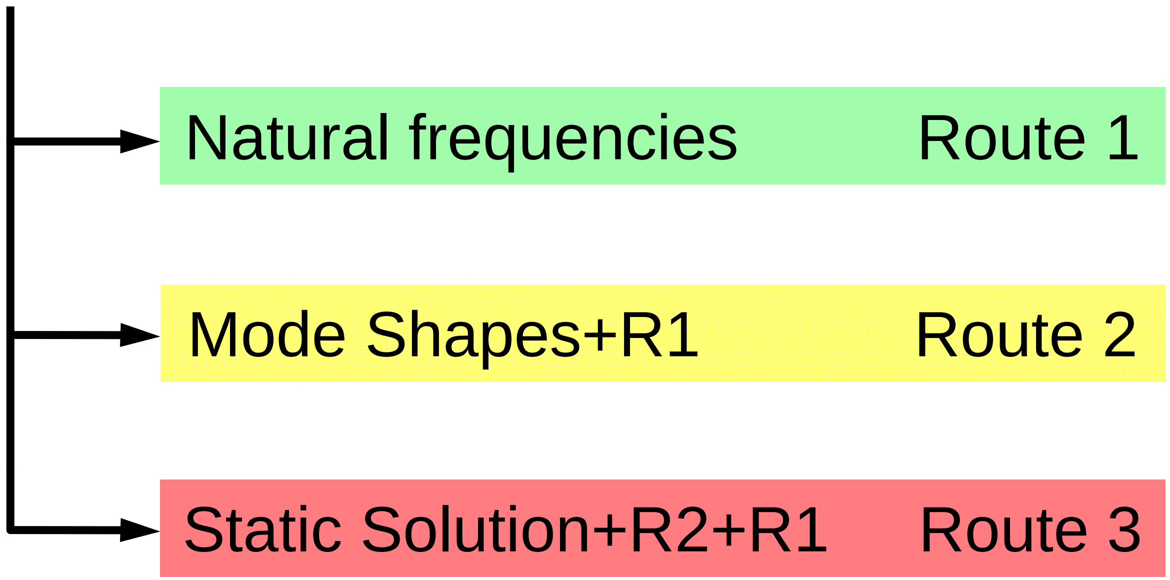
MOTIVATION

• Influence of structural variability on aeroelastic response:

- Natural frequencies — Route 1
- Mode Shapes+R1 — Route 2
- Static Solution

MOTIVATION

• Influence of structural variability on aeroelastic response:



MOTIVATION

• Influence of structural variability on aeroelastic response:

- Natural frequencies — Route 1
- Mode Shapes+R1 — Route 2
- Static Solution+R2+R1 — Route 3

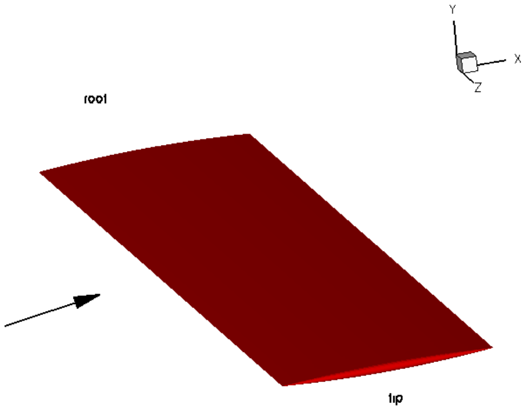
• Previous work has suggested only considering route 1, which allows significant gains in computational efficiency if reduced order models can be built for the aerodynamics.

MOTIVATION

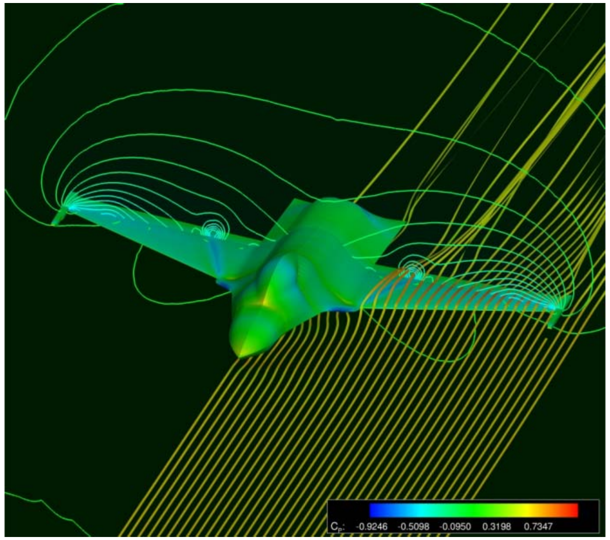
However, neglecting Route 2 and 3 can give misleading results for flutter onset prediction.

TEST CASES

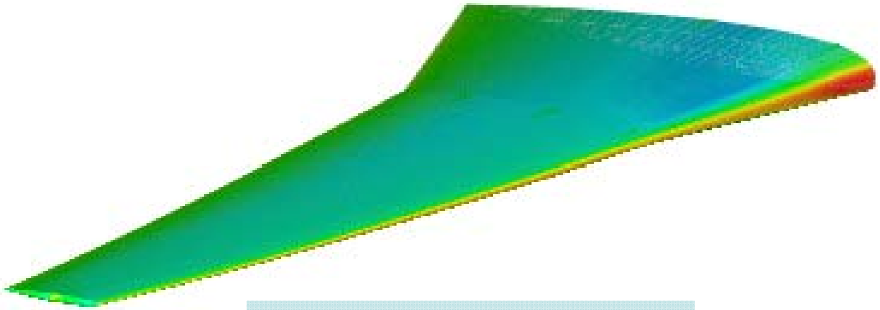
TEST CASES



Goland Wing+

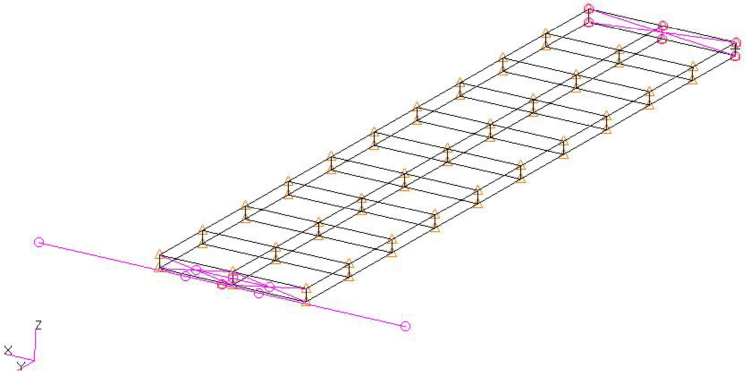


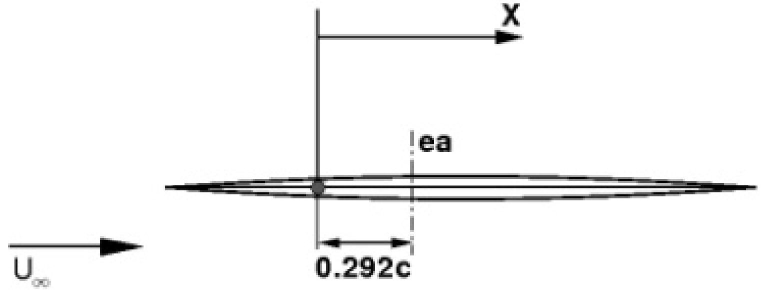
Open Source Fighter

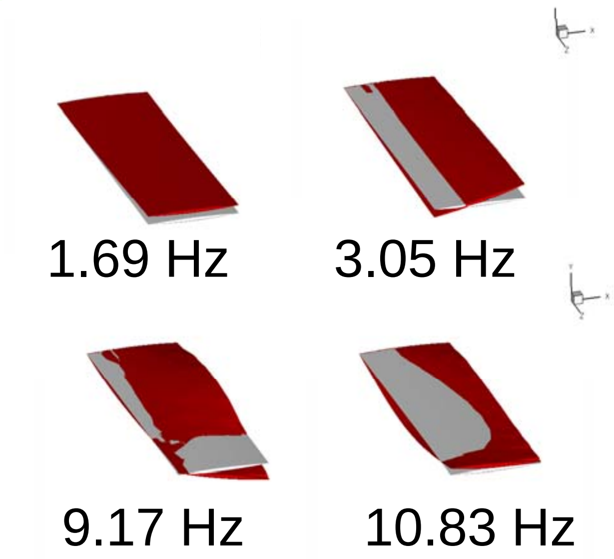


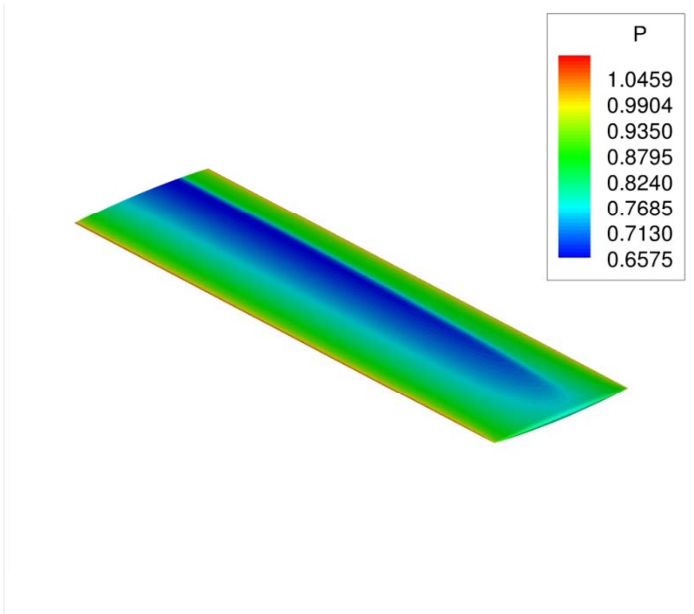
MDO Wing

GOLAND WING - STORE



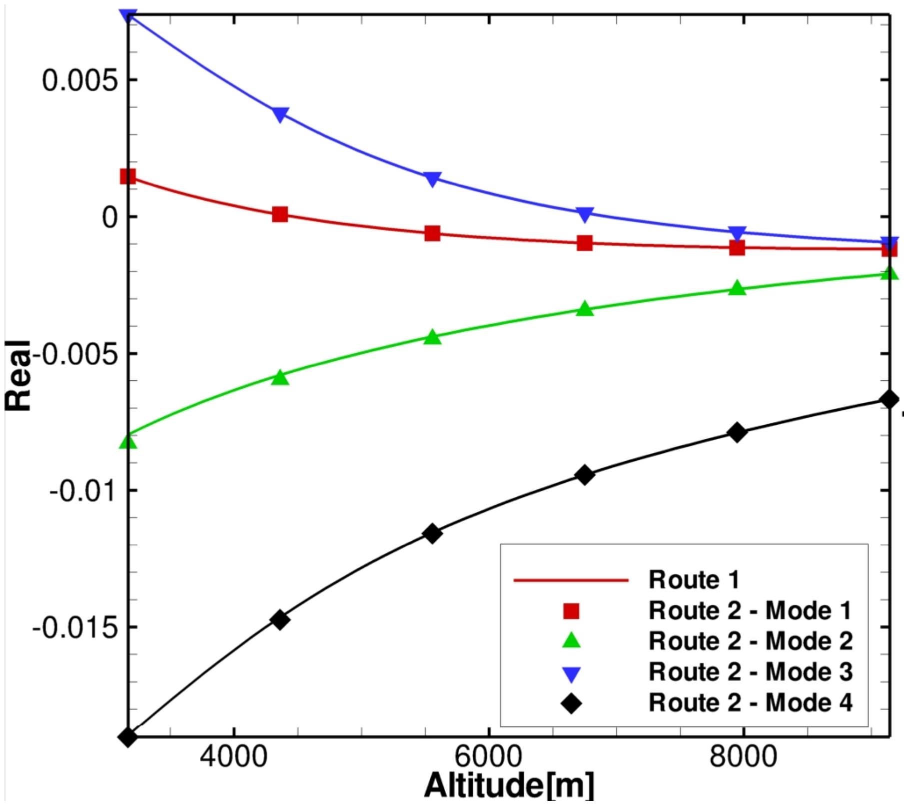


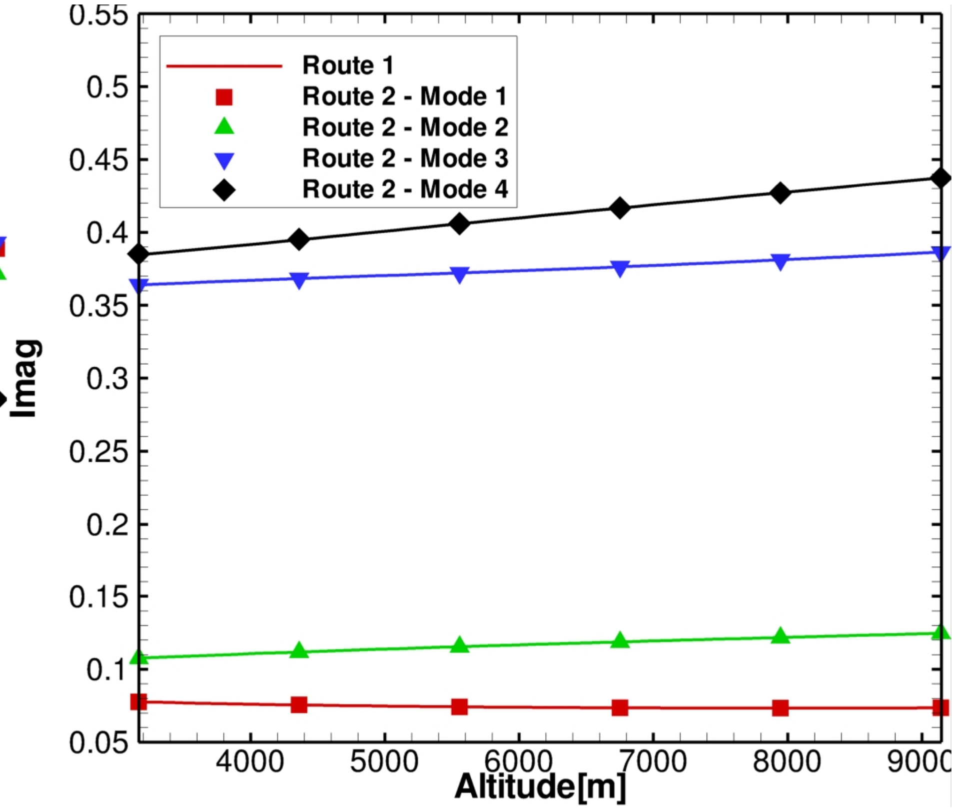




GOLAND WING - STORE

- M=0.91 α=0°





GOLAND WING - STORE

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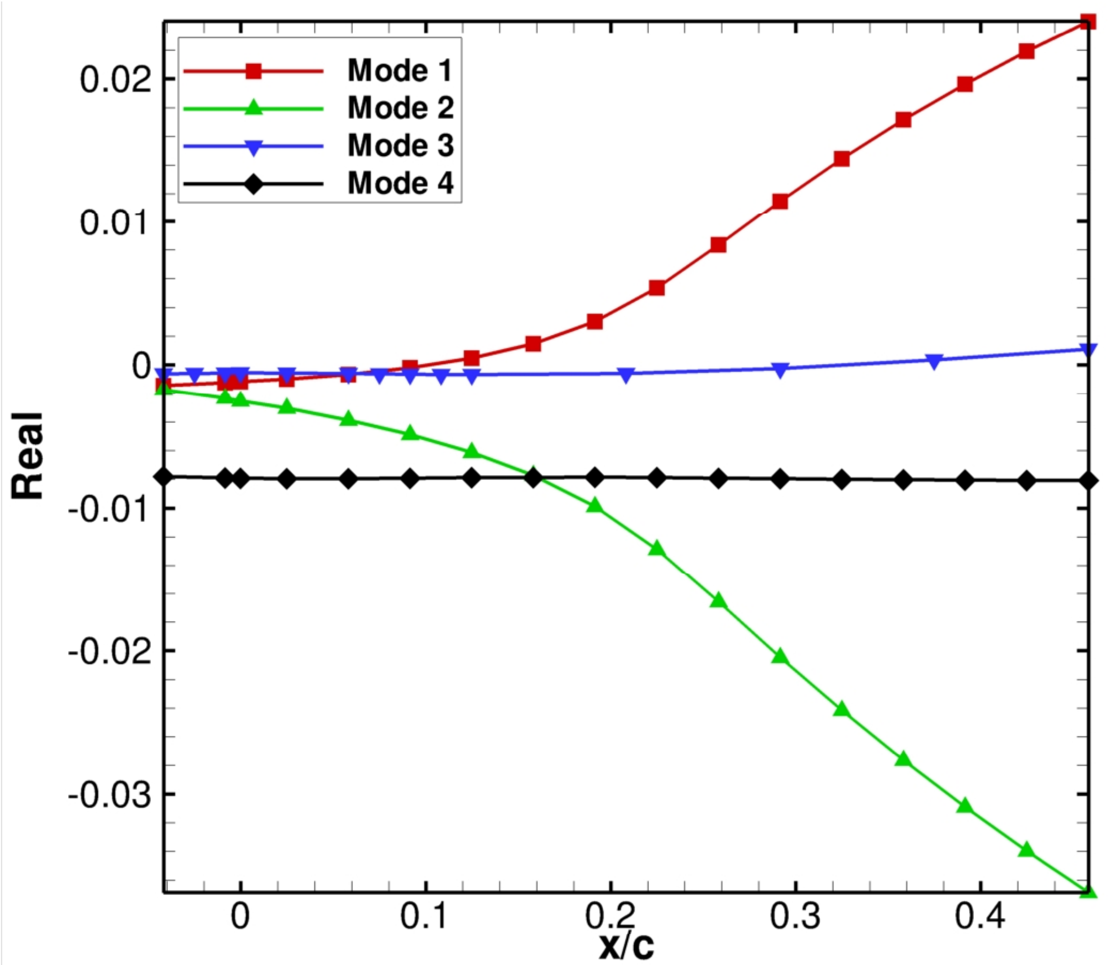
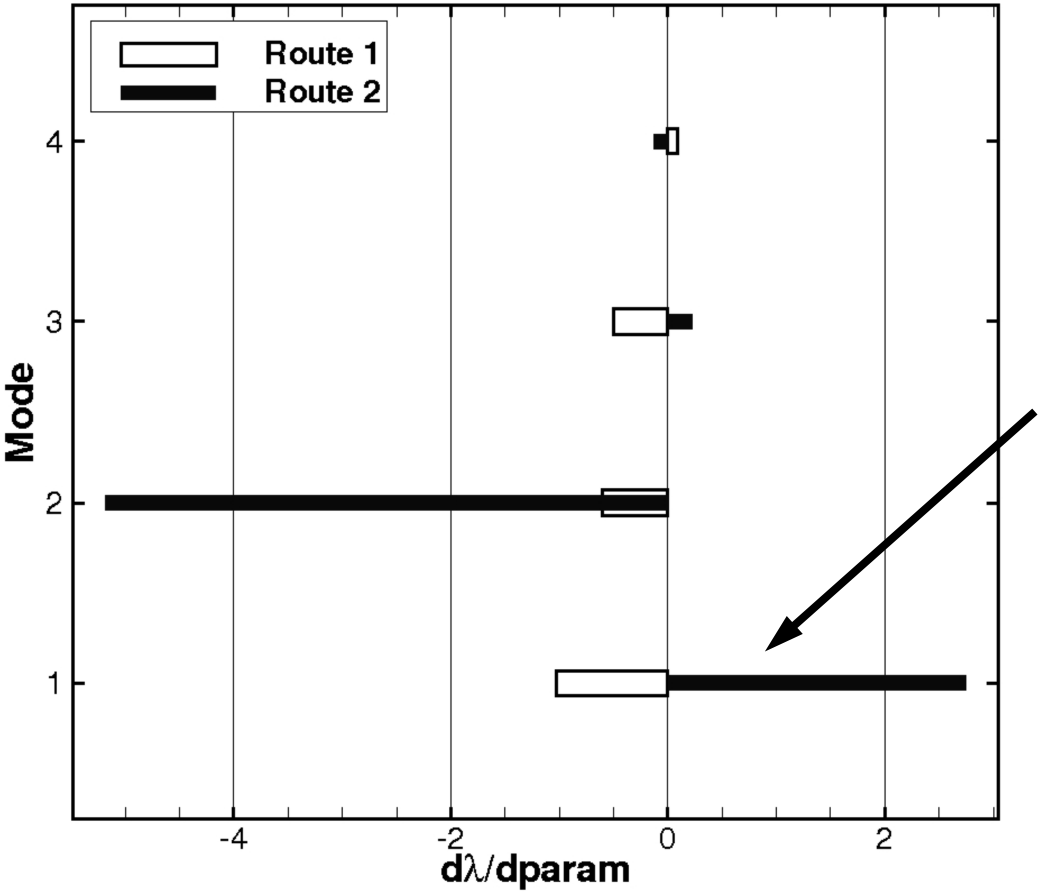
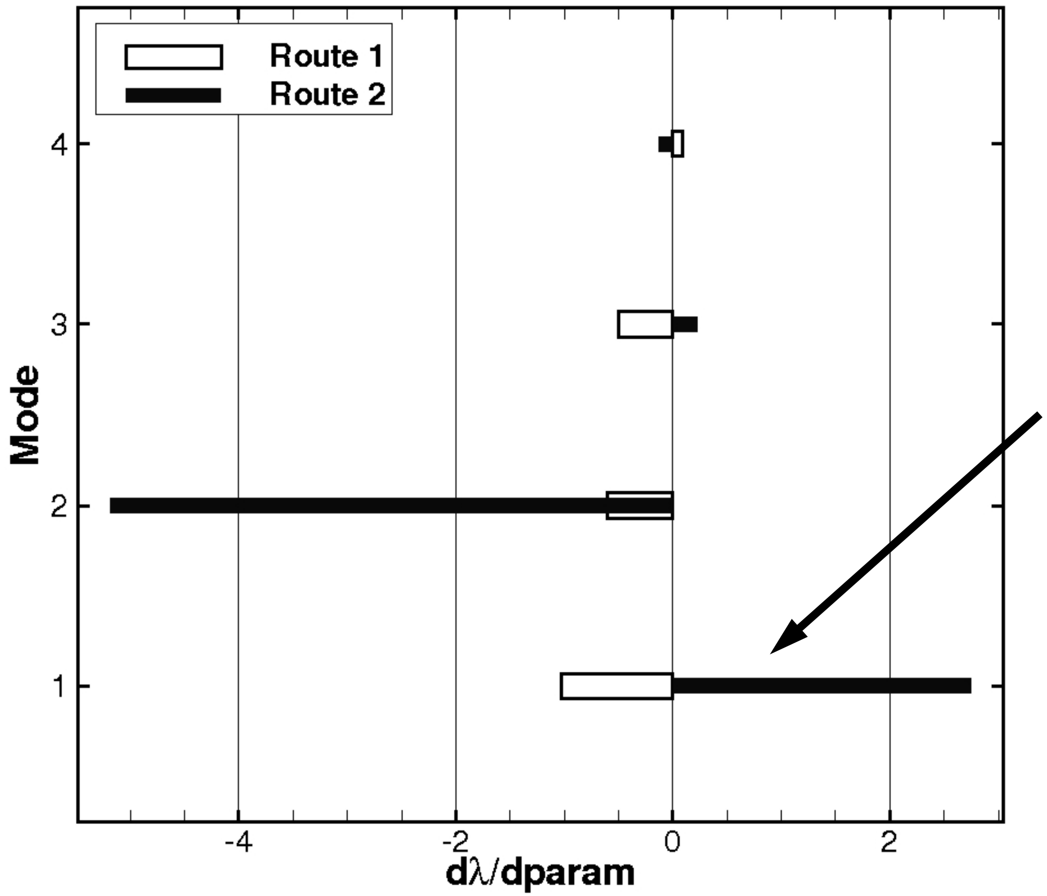


GOLAND WING - STORE



Note oposite signs in mode 1 sensitivity

GOLAND WING - STORE

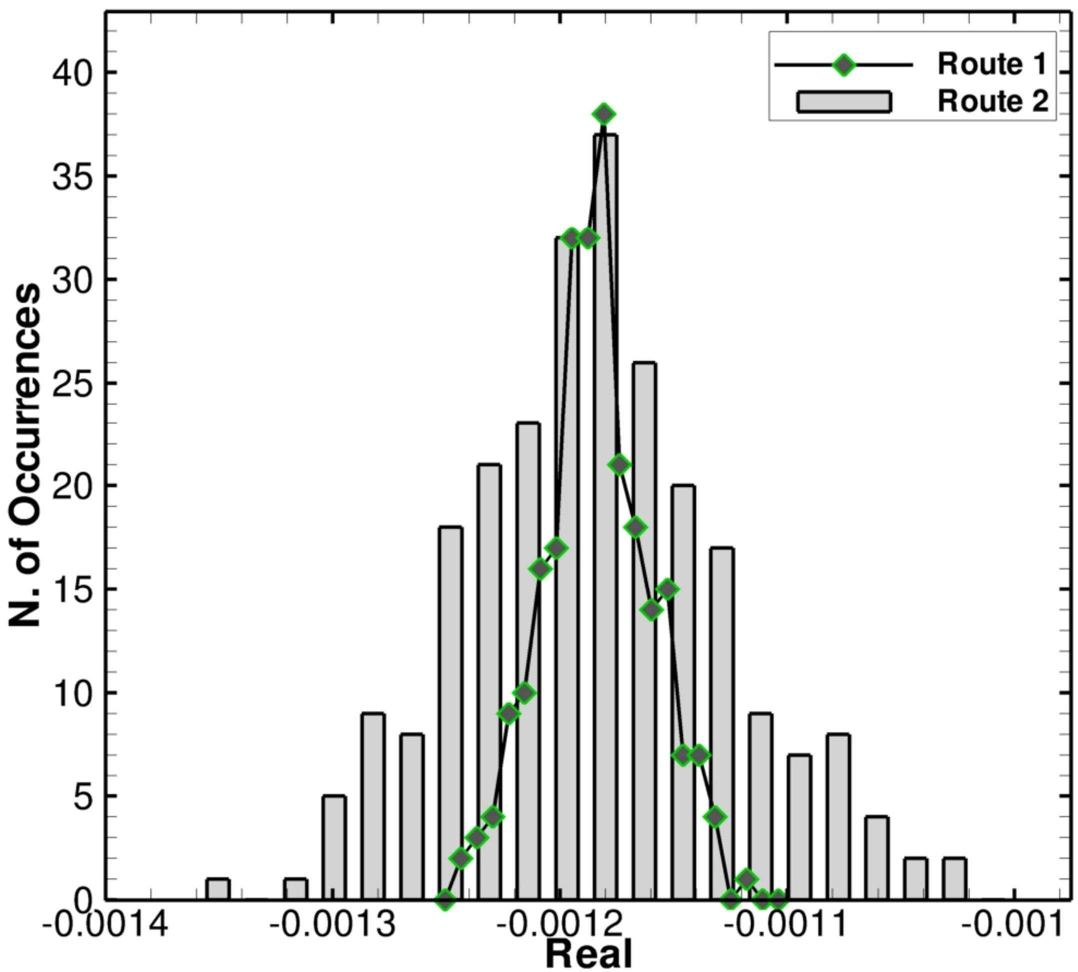


Note oposite signs in mode 1 sensitivity

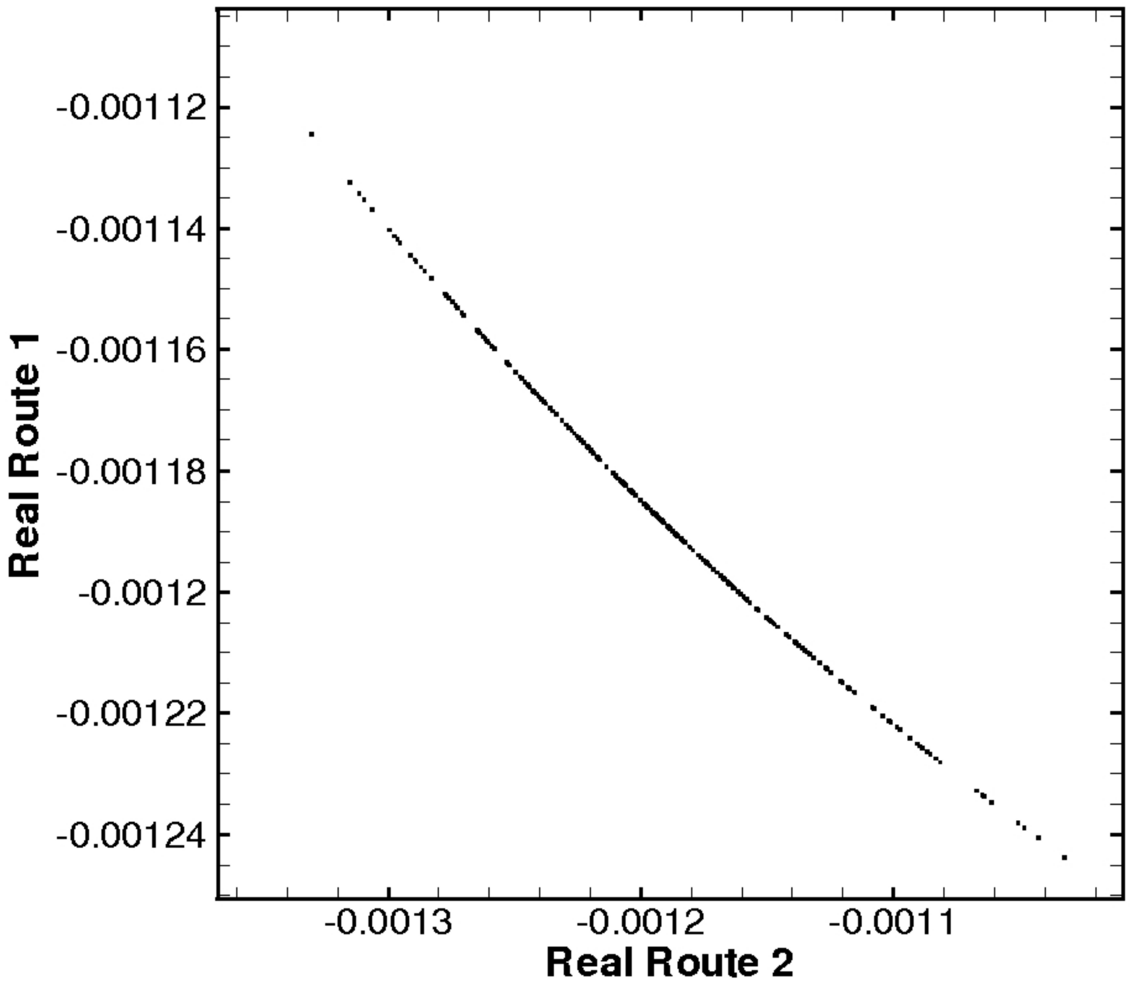
- Normal distribution is assumed for the store location with COVs of 2.8 %

GOLAND WING - STORE

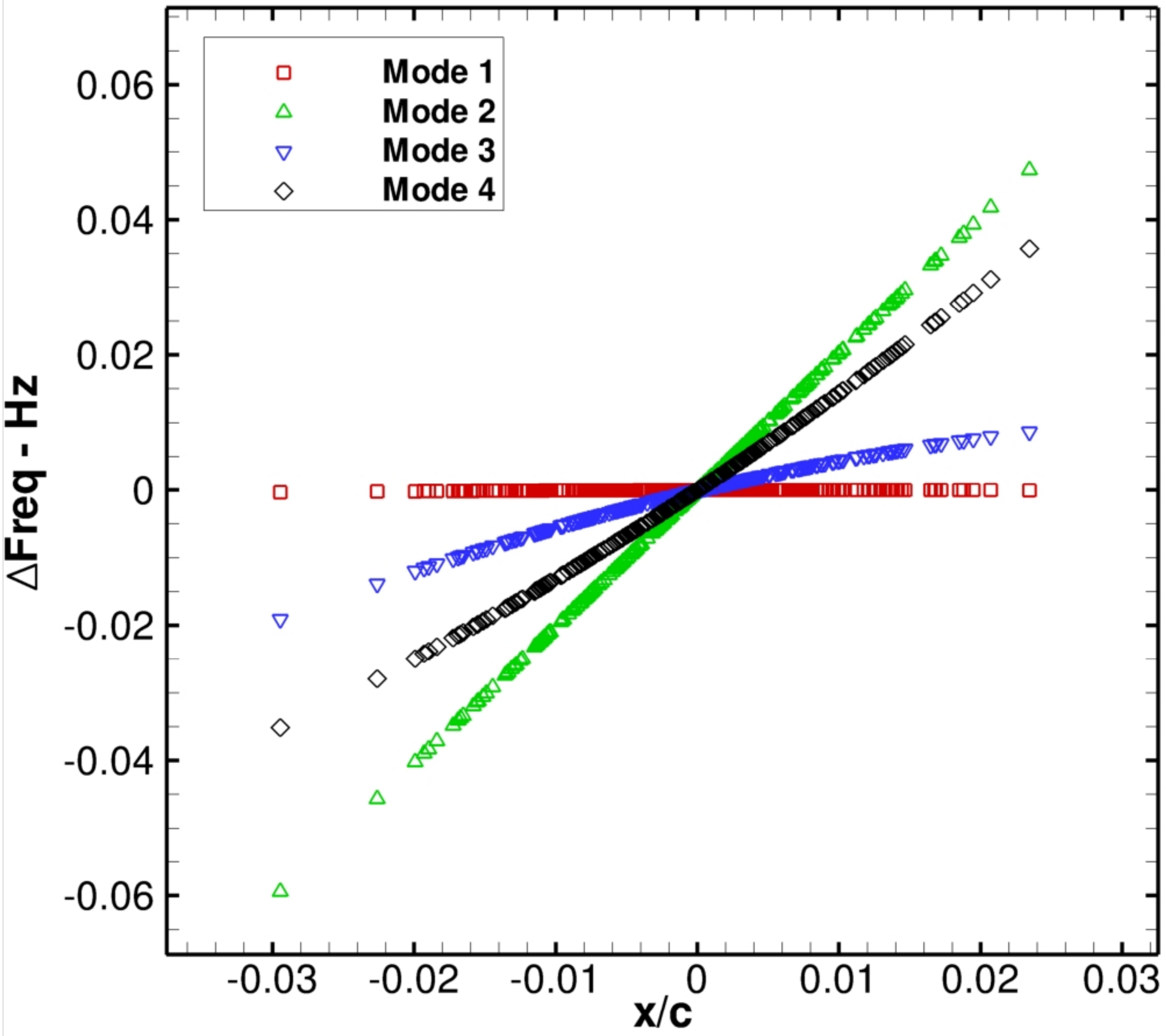
- M=0.91 α=0° , mode 1



GOLAND WING - STORE

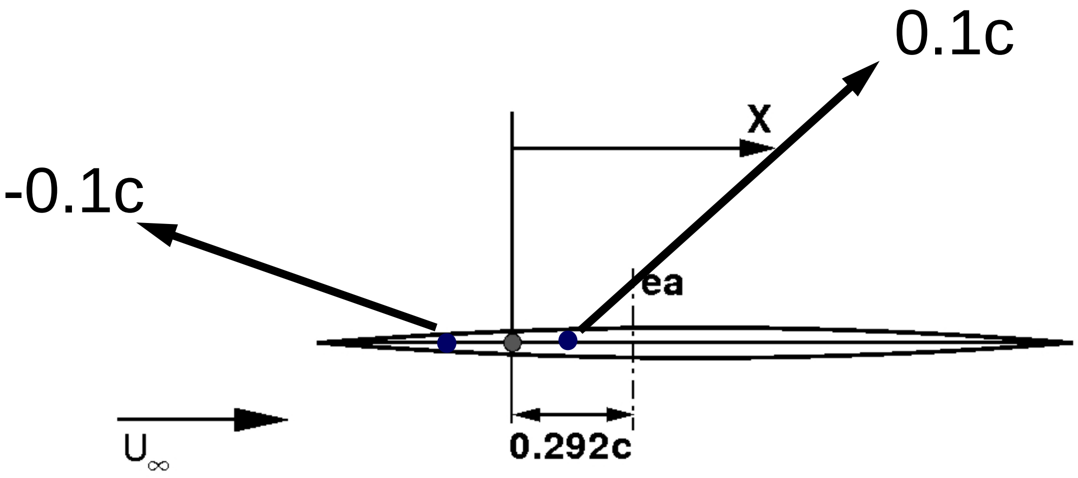


GOLAND WING - STORE

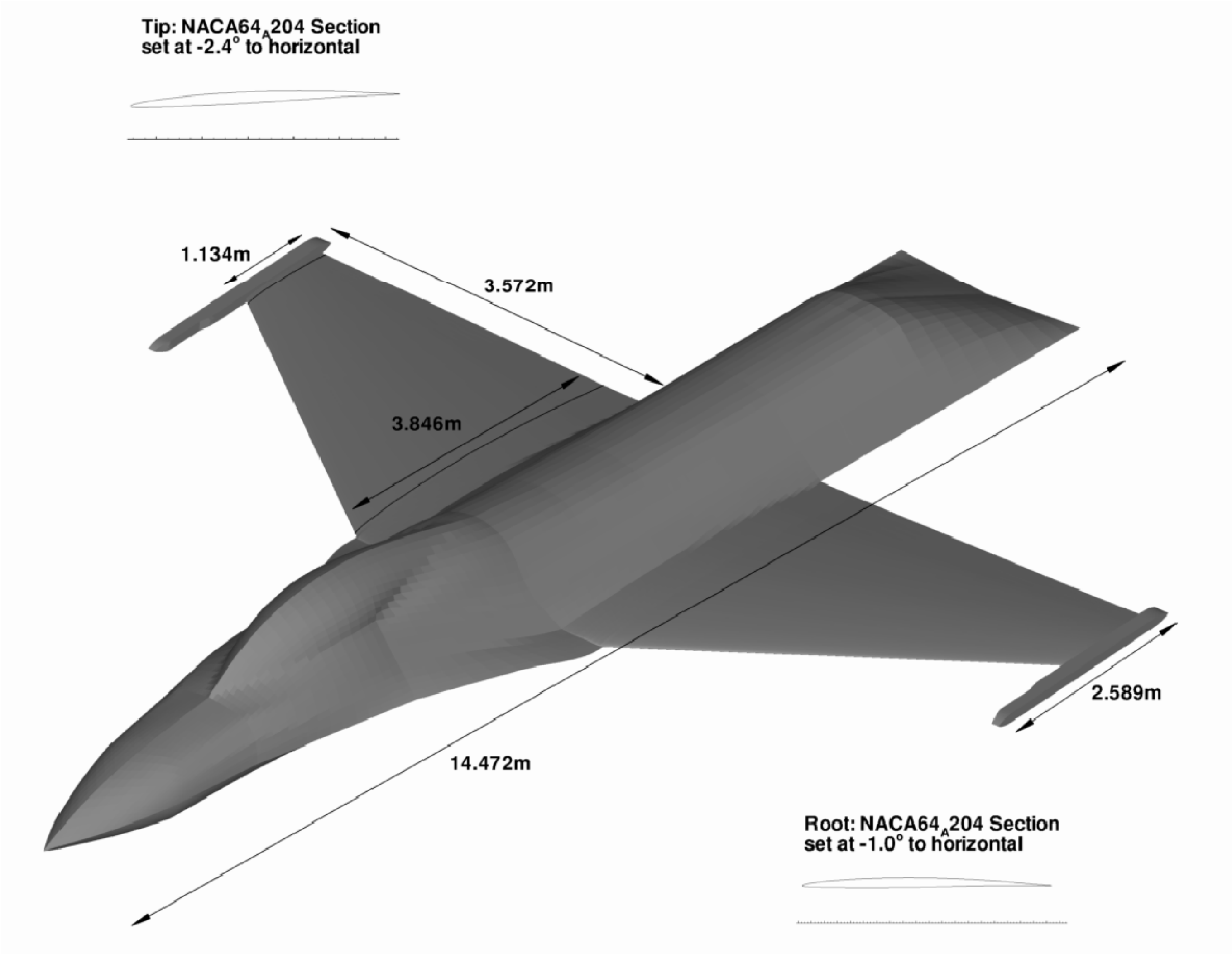


GOLAND WING - STORE

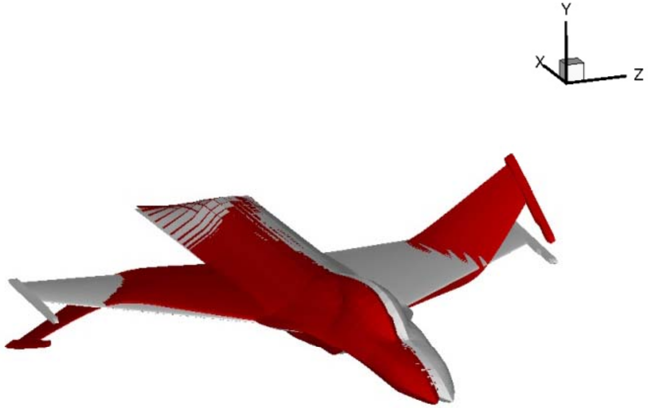
- M=0.91 α=0°



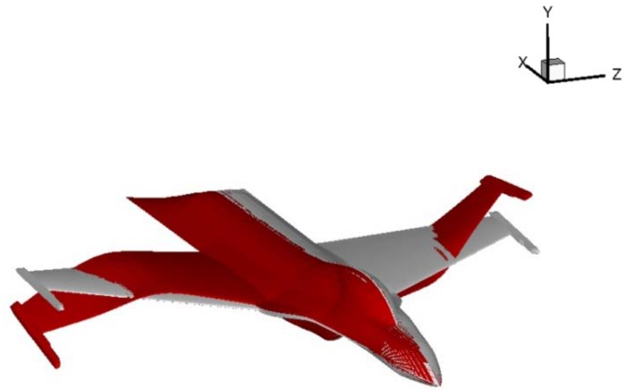
Open Source Fighter



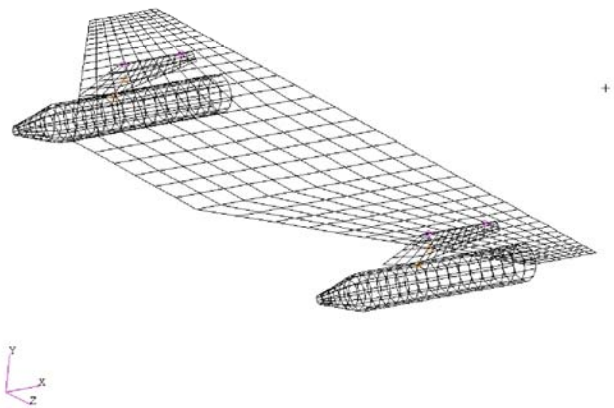
Generic Fighter

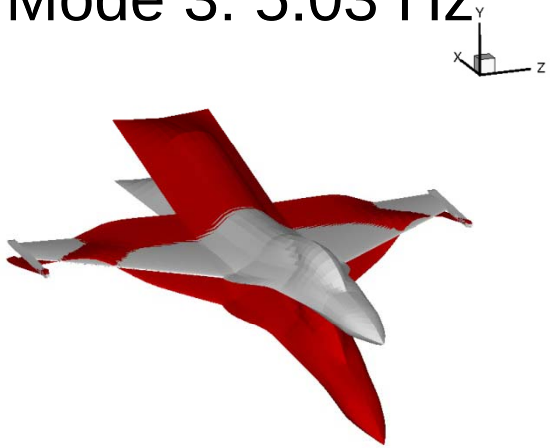


Mode 2: 4.48 Hz



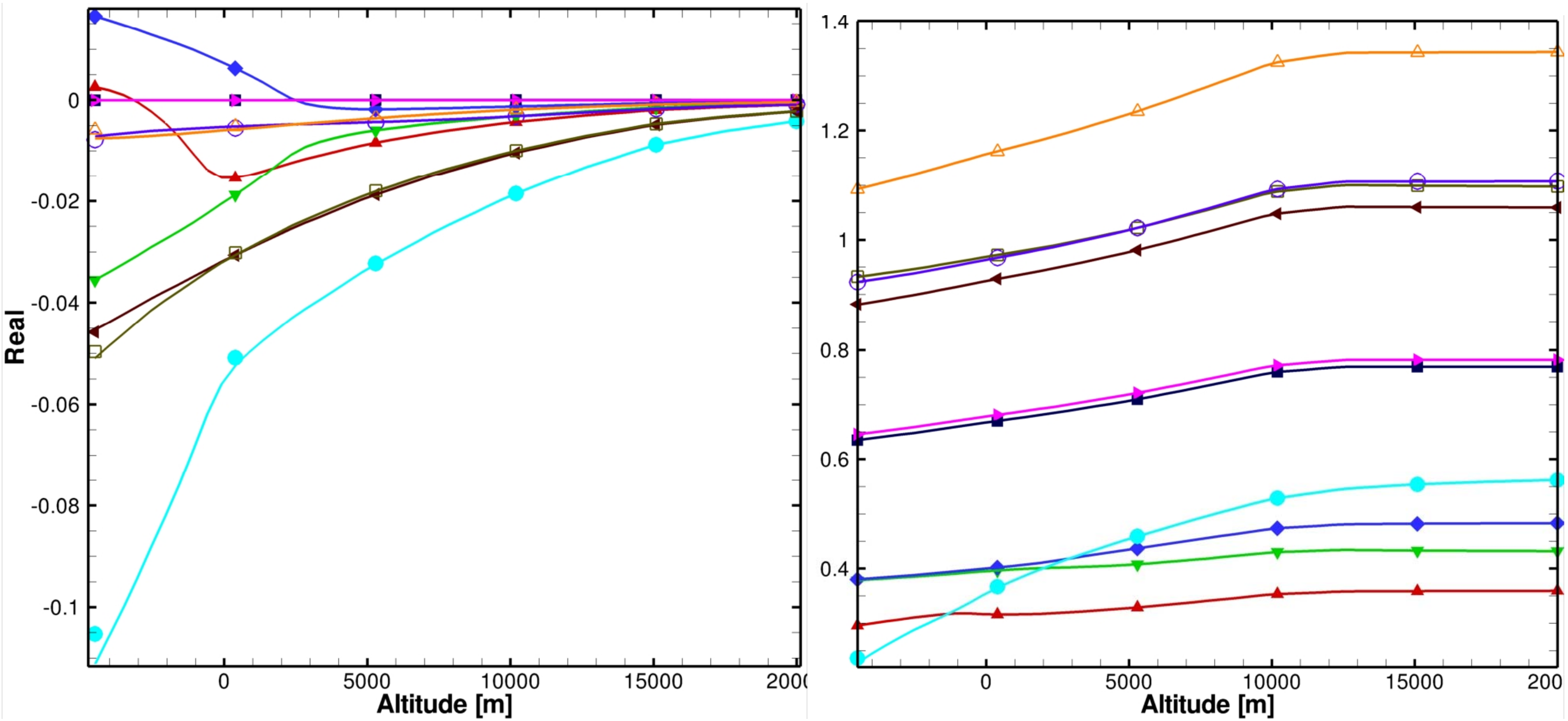
Mode 3: 5.03 Hz





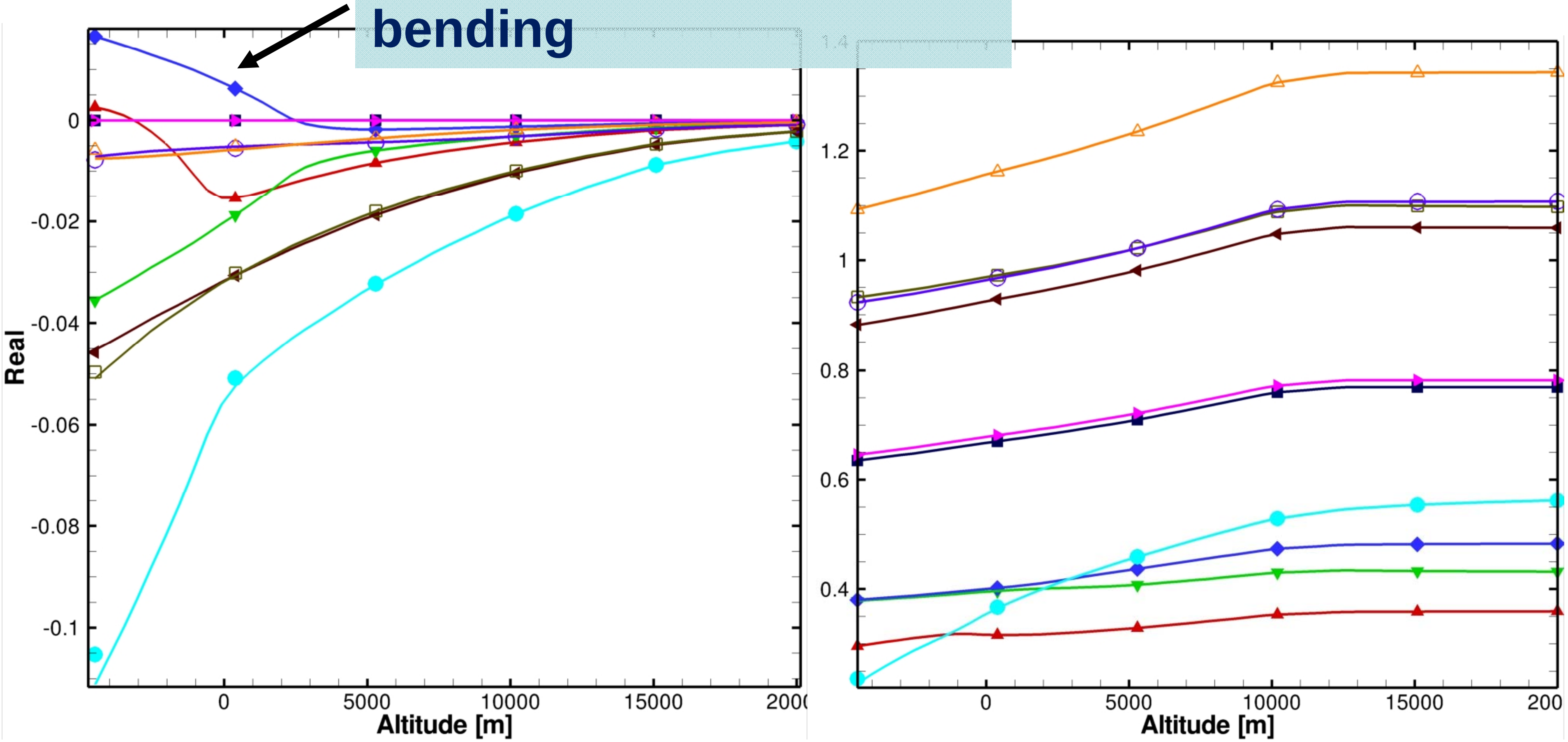
Mode 4: 5.92 Hz

Generic Fighter

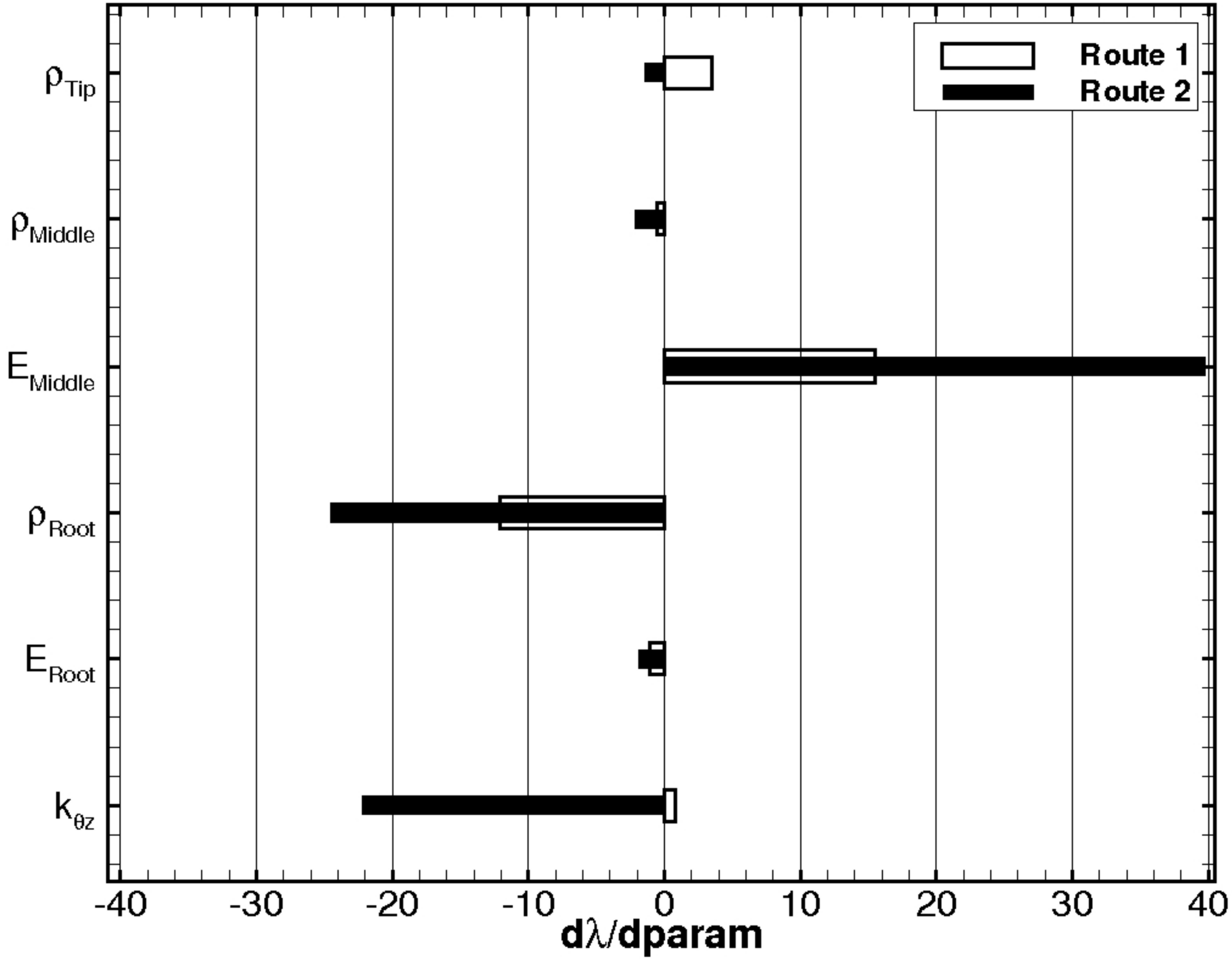


Generic Fighter

Mode 3 – Antisymmetric bending

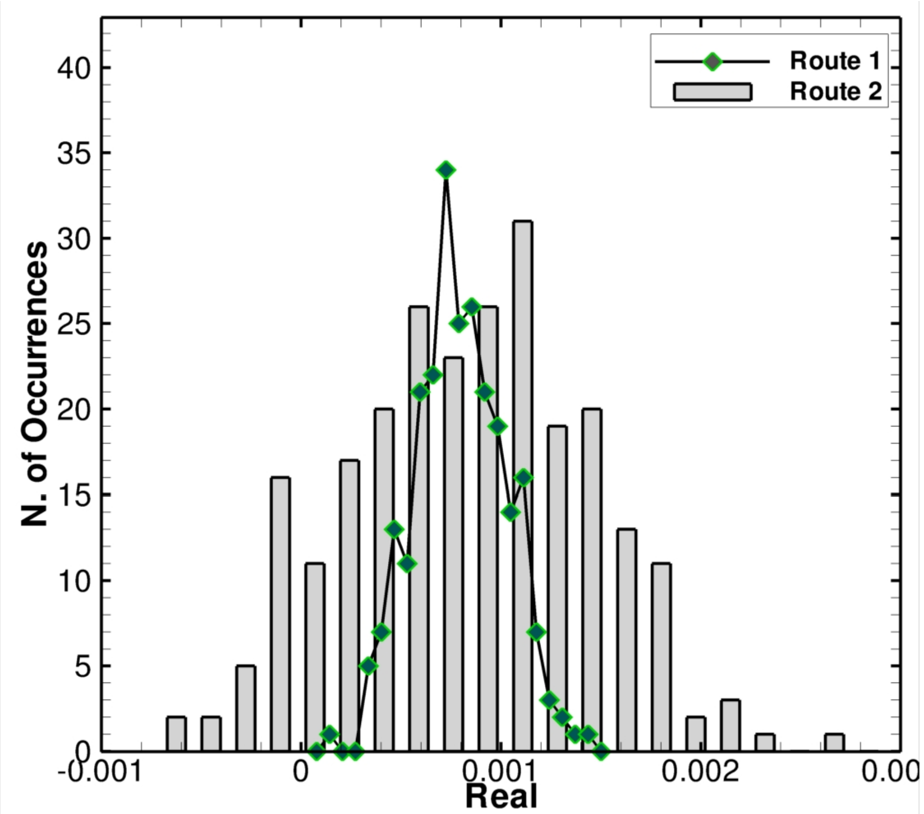


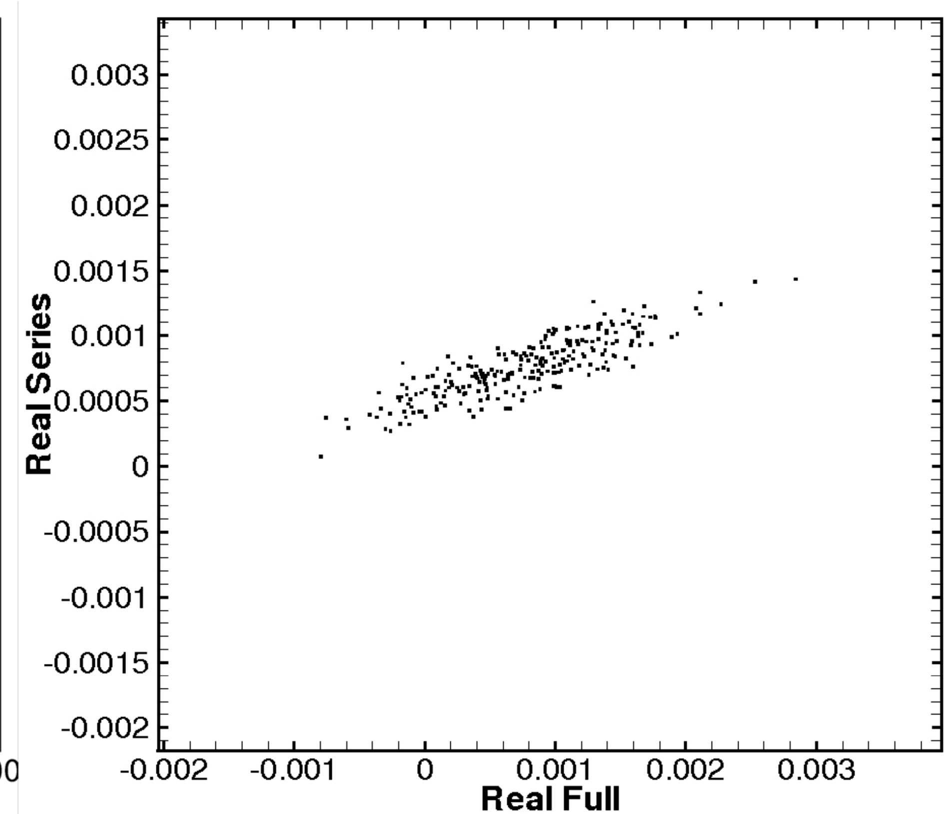
Generic Fighter



Generic Fighter

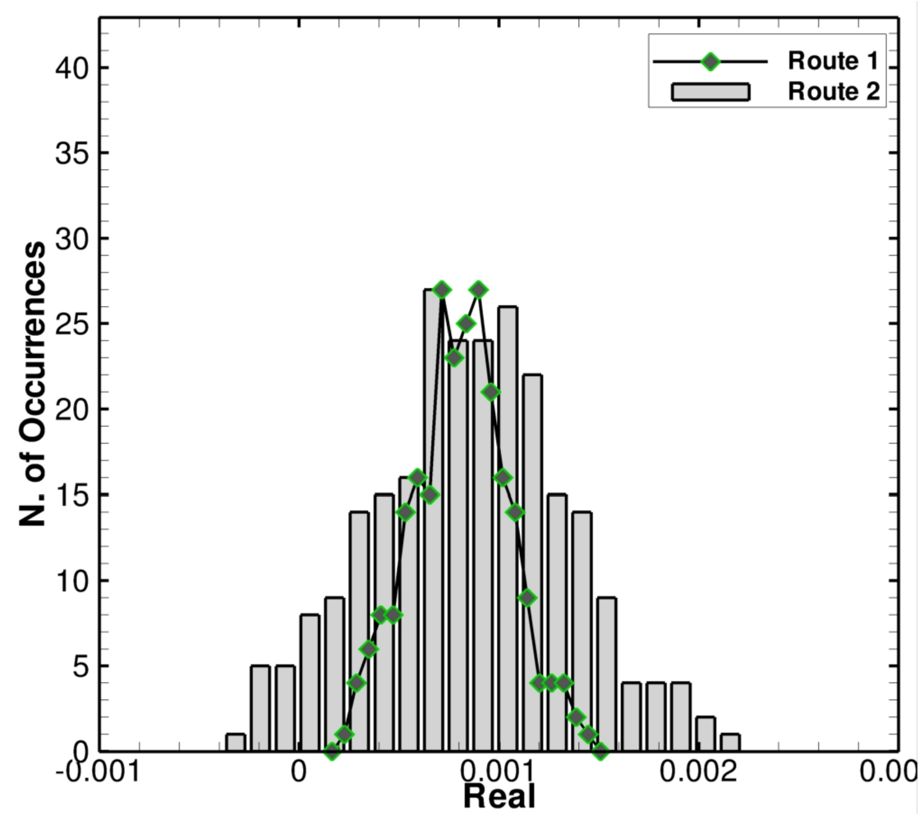
6 parameters

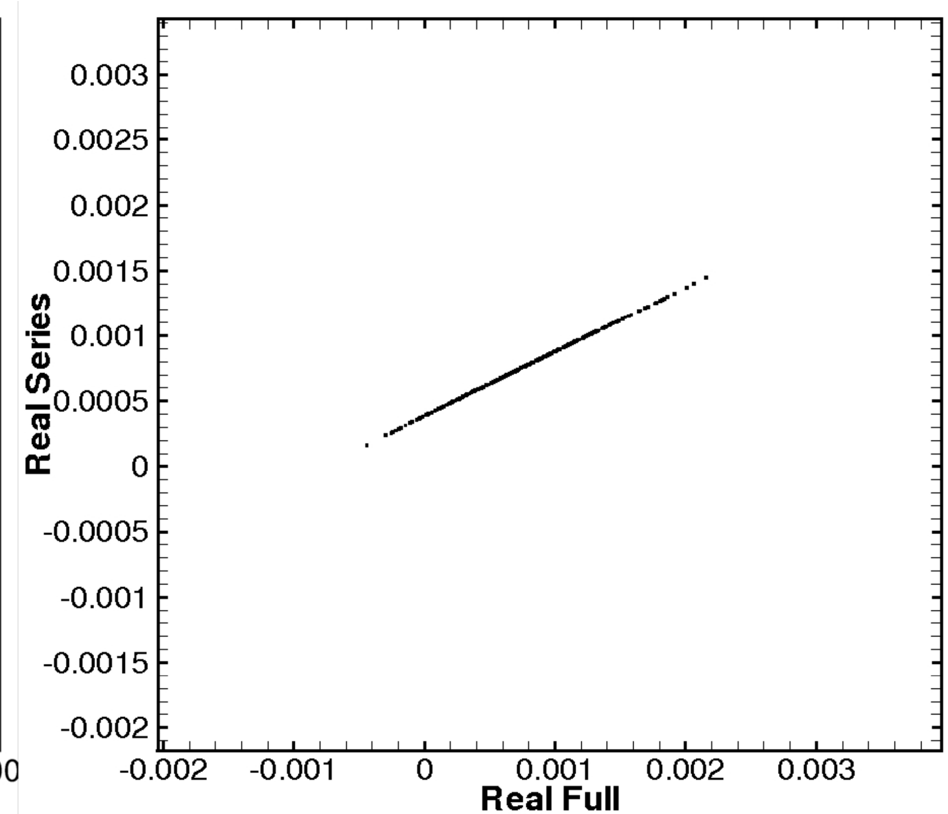




Generic Fighter

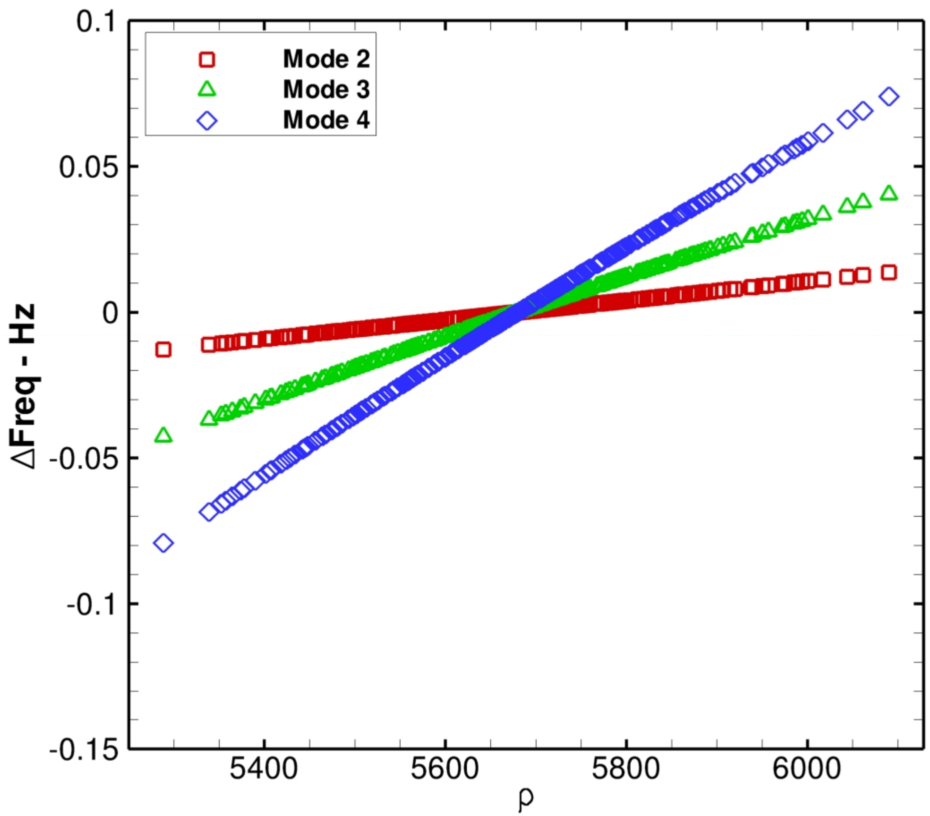
Wing root density

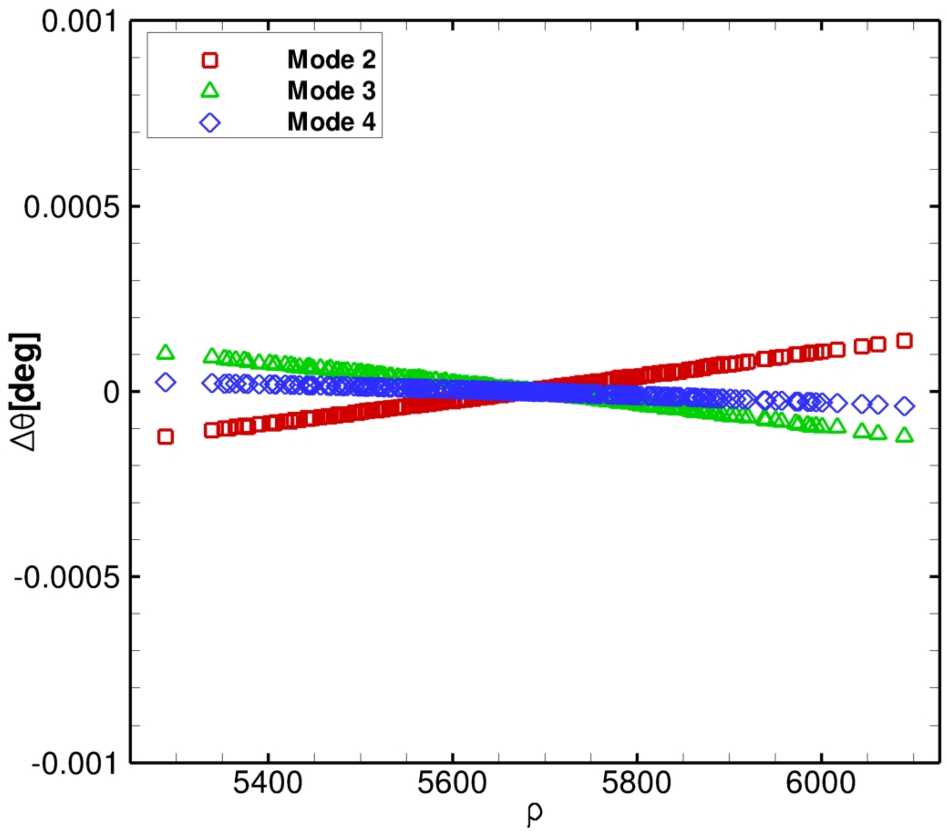




Generic Fighter

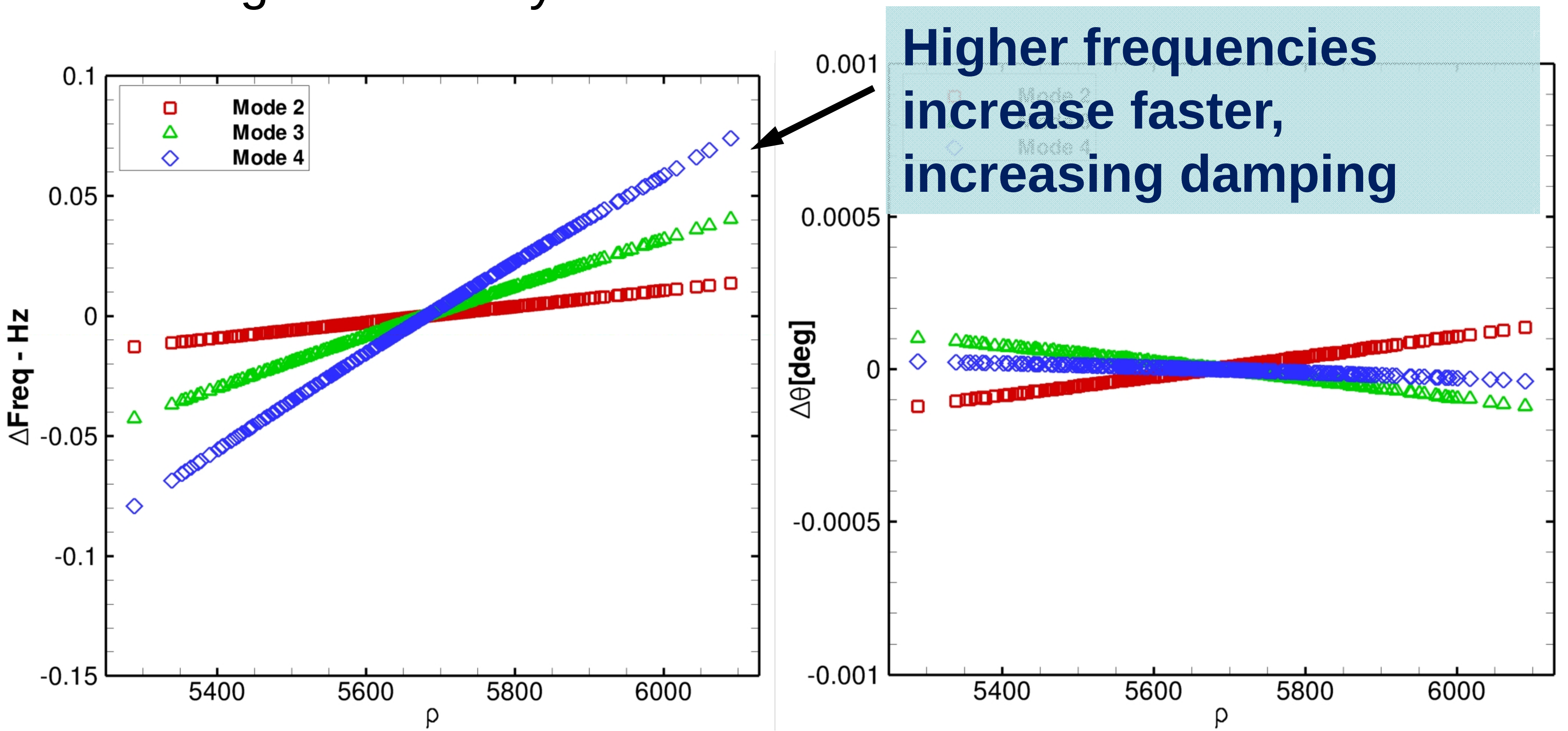
Wing root density





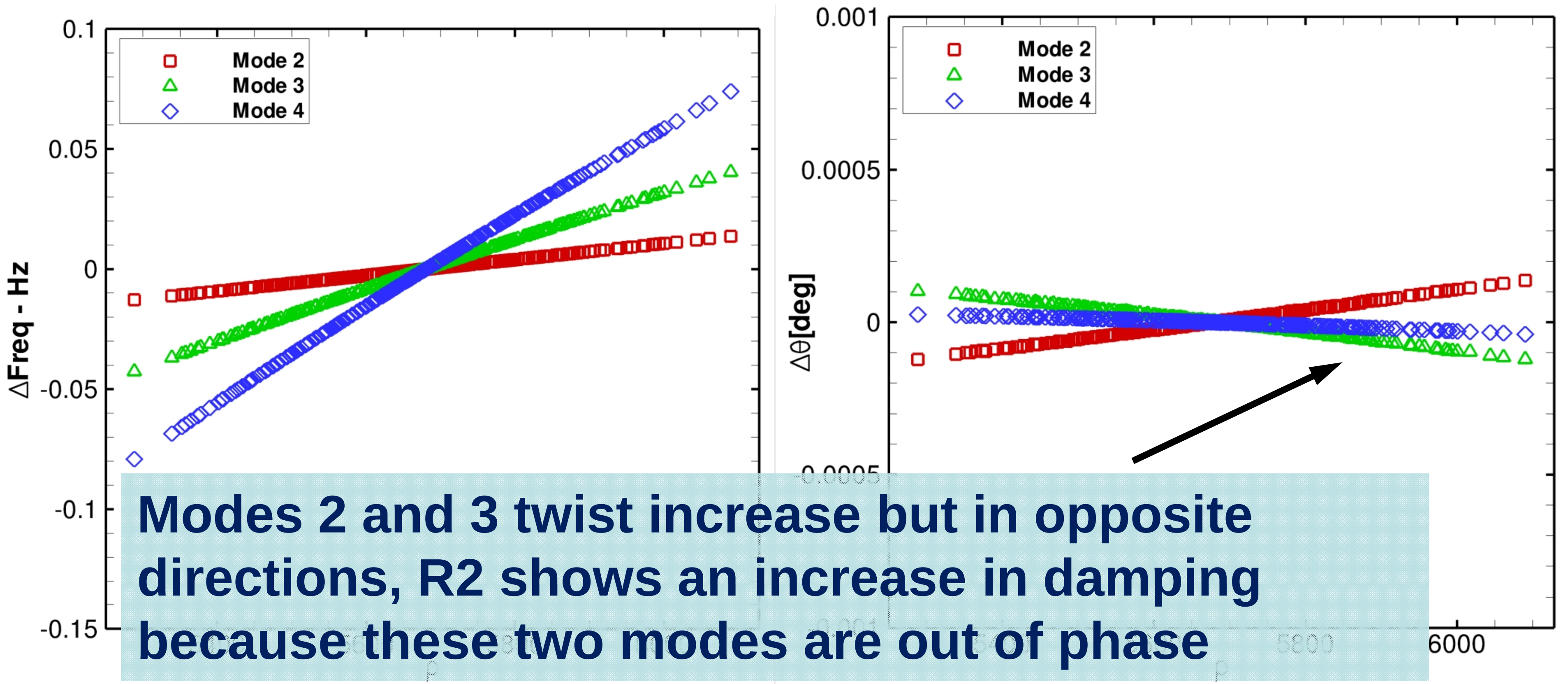
Generic Fighter

Wing root density



Generic Fighter

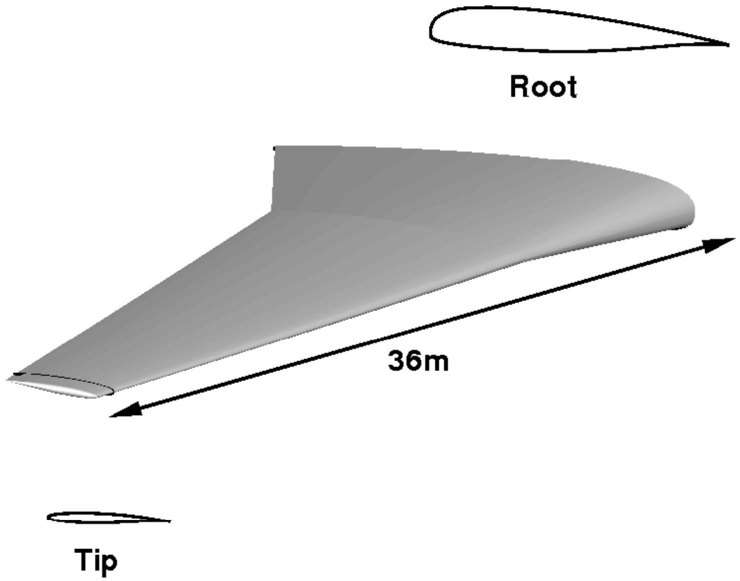
Wing root density



Modes 2 and 3 twist increase but in opposite directions, R2 shows an increase in damping because these two modes are out of phase

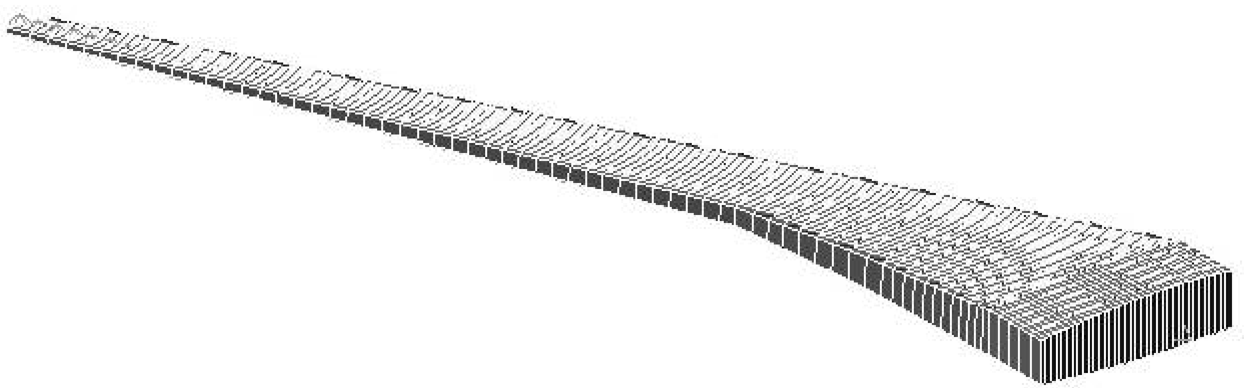
MDO Wing

- Representative of a typical transport wing
- 36m Span



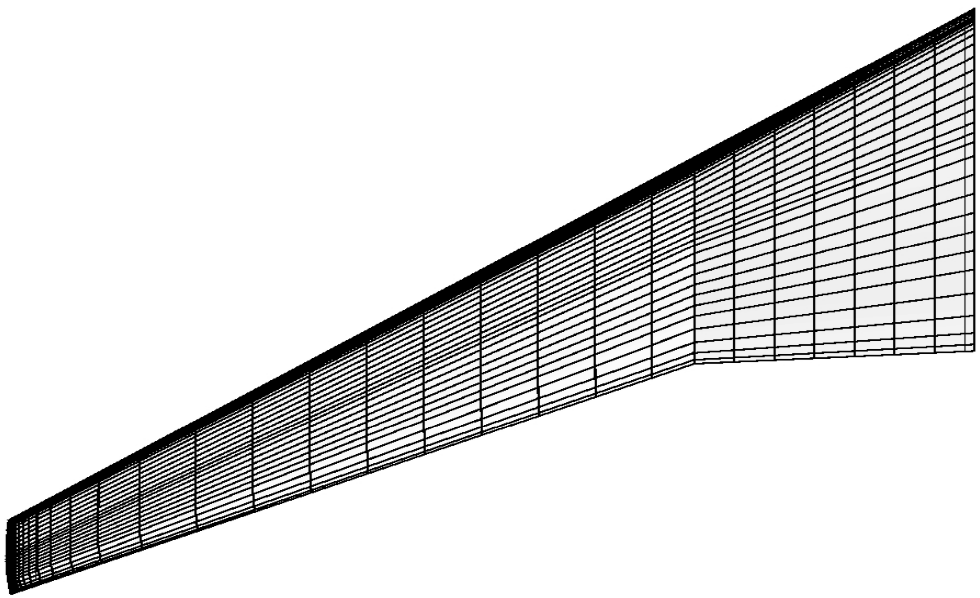
MDO Wing

- Representative of a typical transport wing
- 36m Span
- 8 Modes retained for analysis



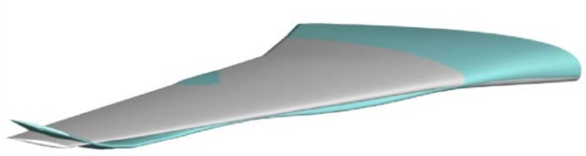
MDO Wing

- Representative of a typical transport wing
- 36m Span
- 8 Modes retained for analysis
- Mach 0.85; α=1°

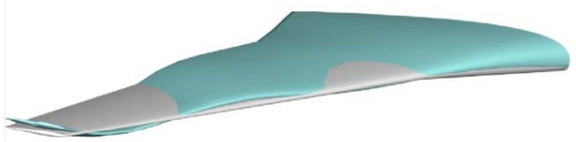


MDO Wing

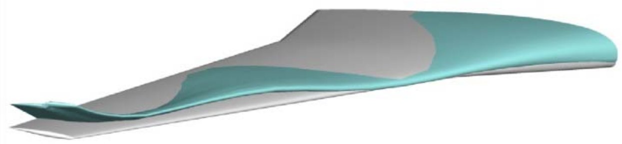
- Representative of a typical transport wing
- 36m Span
- 8 Modes retained for analysis
- Mach 0.85; $\alpha=1^{\circ}$
- The wing density, Young's modulus and Shear modulus are considered uncertain (Guassian distribution with COVs 2.8%).



0.844 Hz

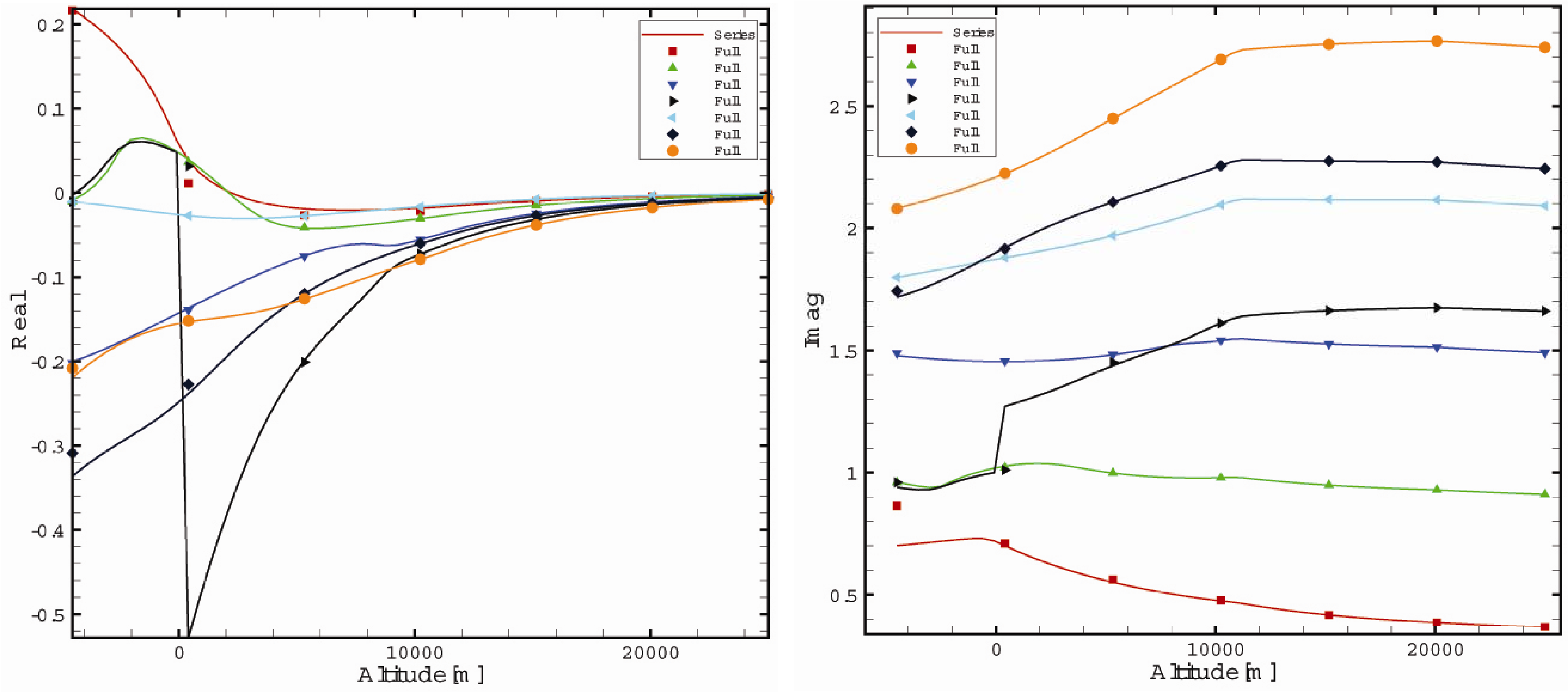


2.162 Hz

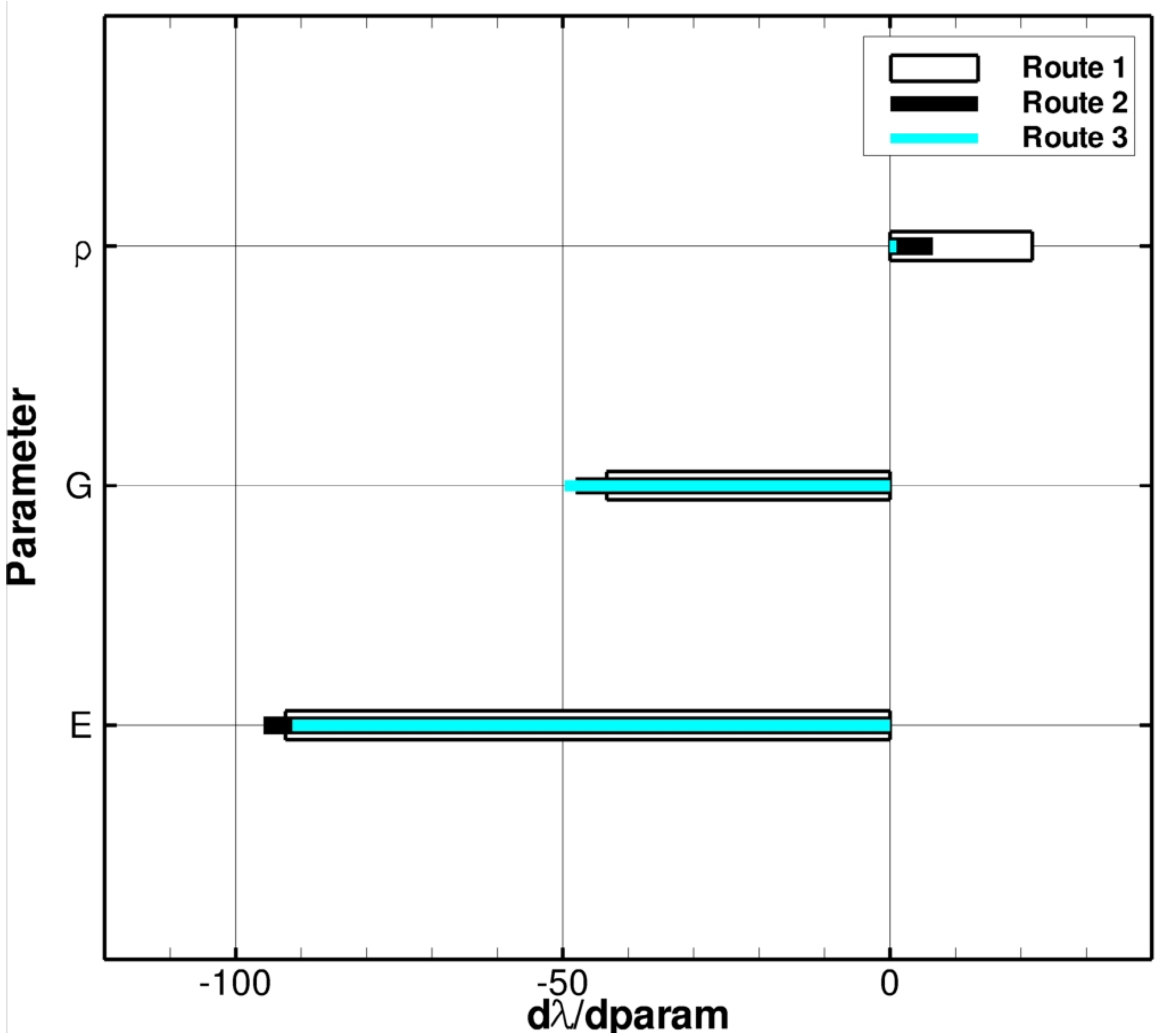


3.989 Hz

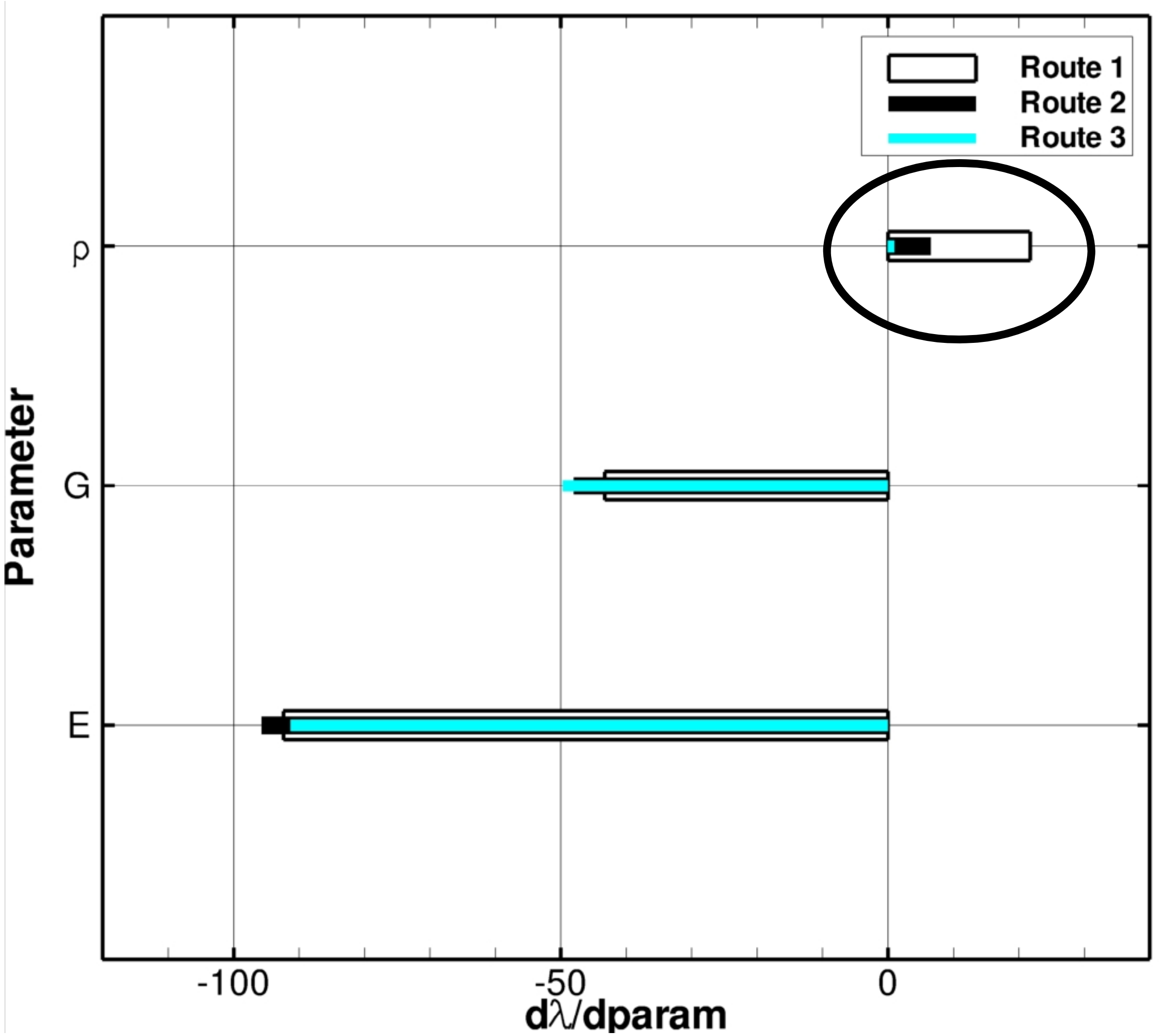
MDO Wing


Series
Full
Real
Altitude [m]
Imag
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
10000
20000
2.5
2
1.5
1
0.5

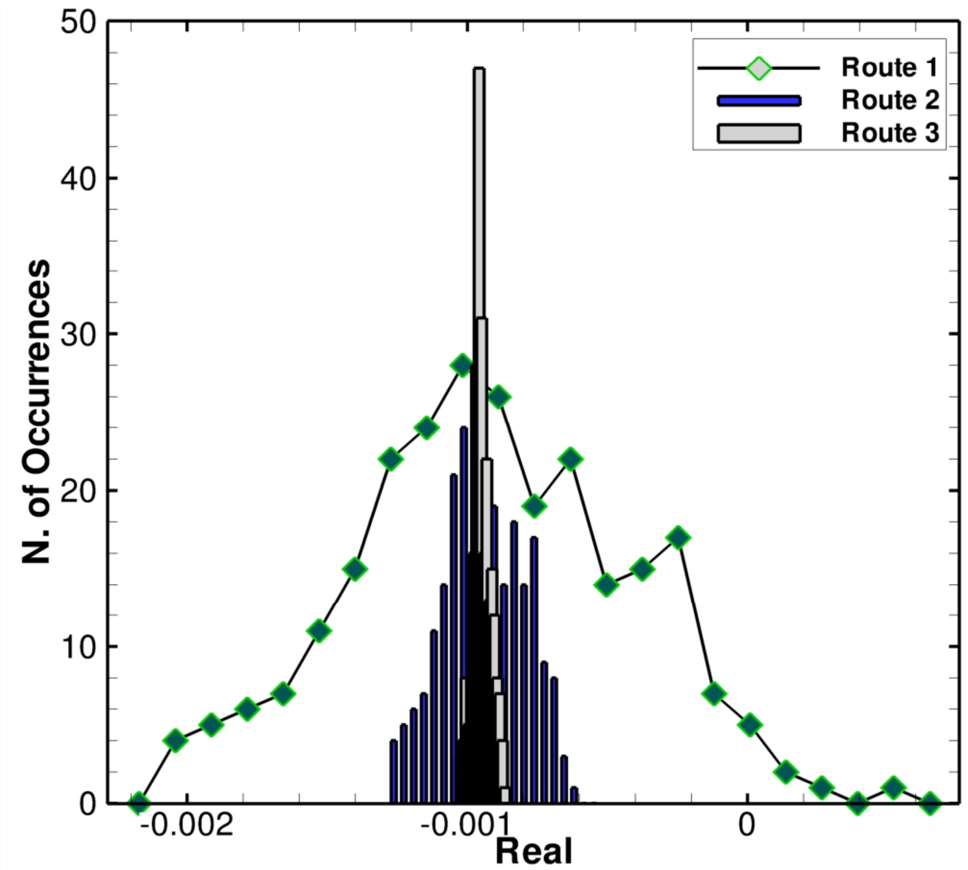
MDO Wing

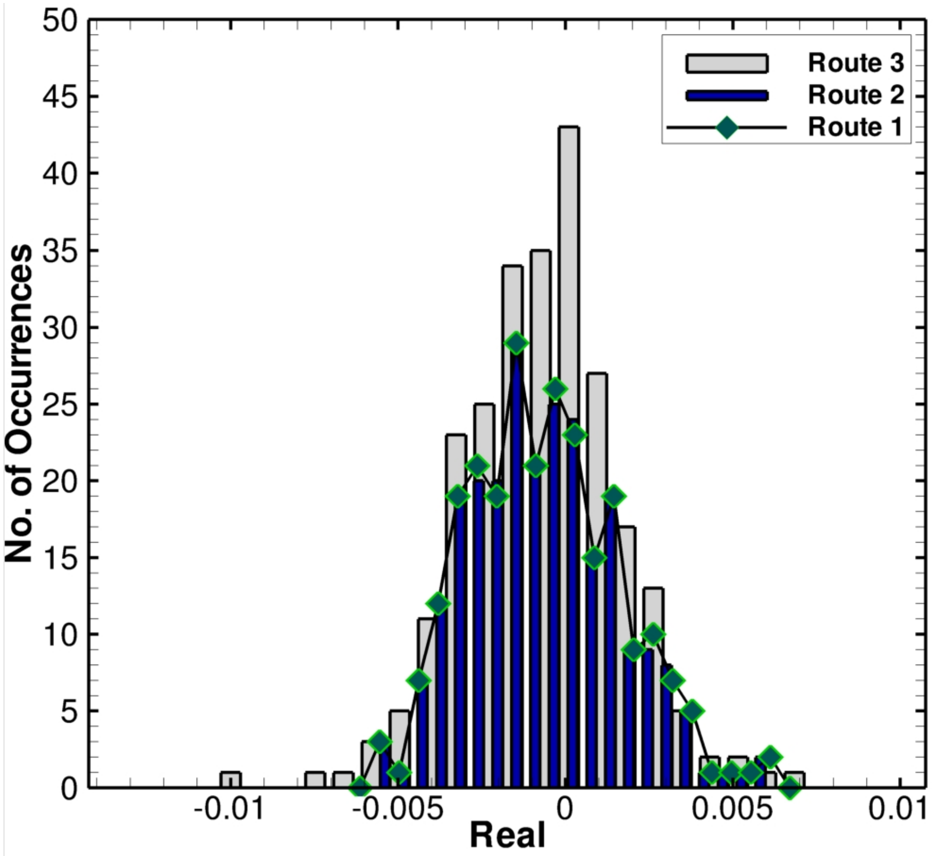


MDO Wing



MDO Wing

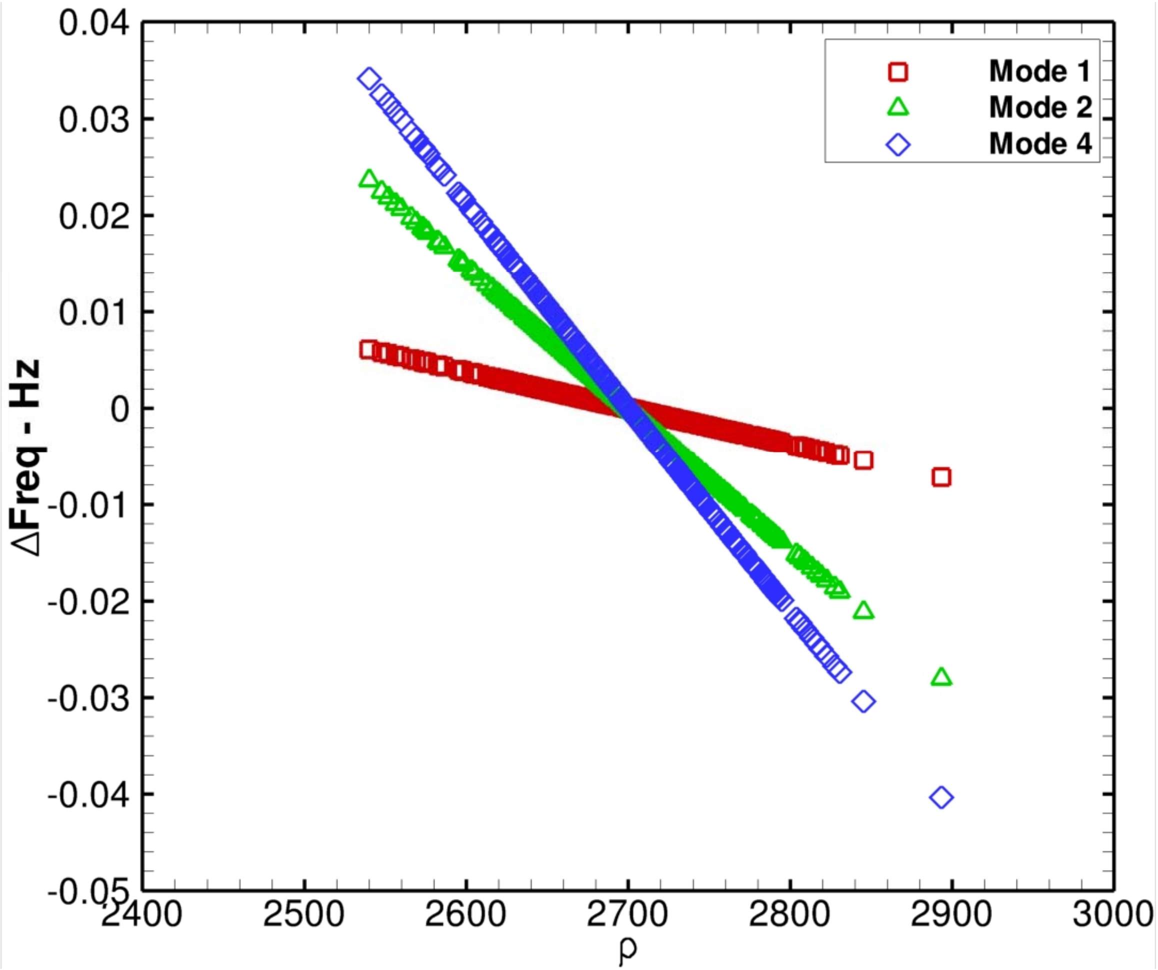




Varying material density

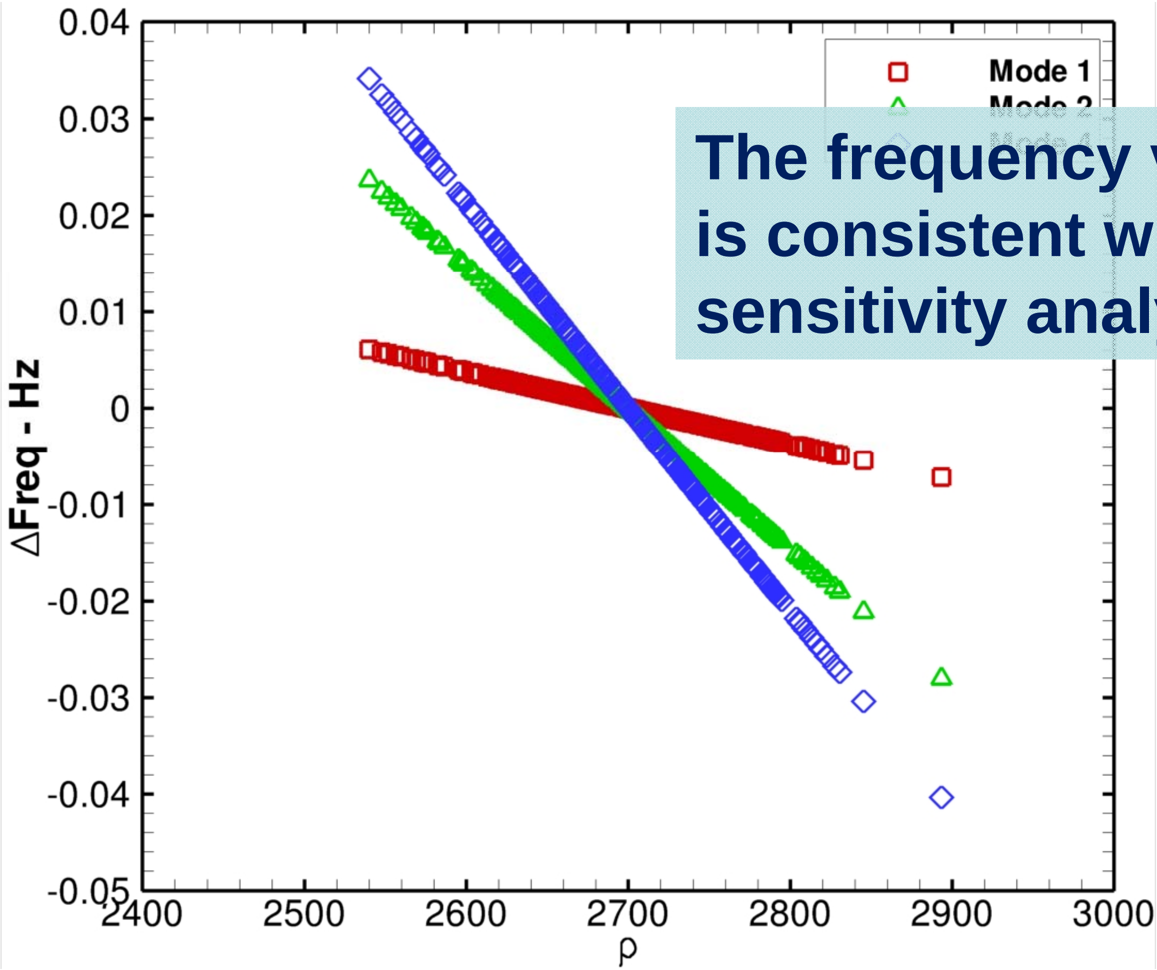
Varying Young's modulus

MDO Wing



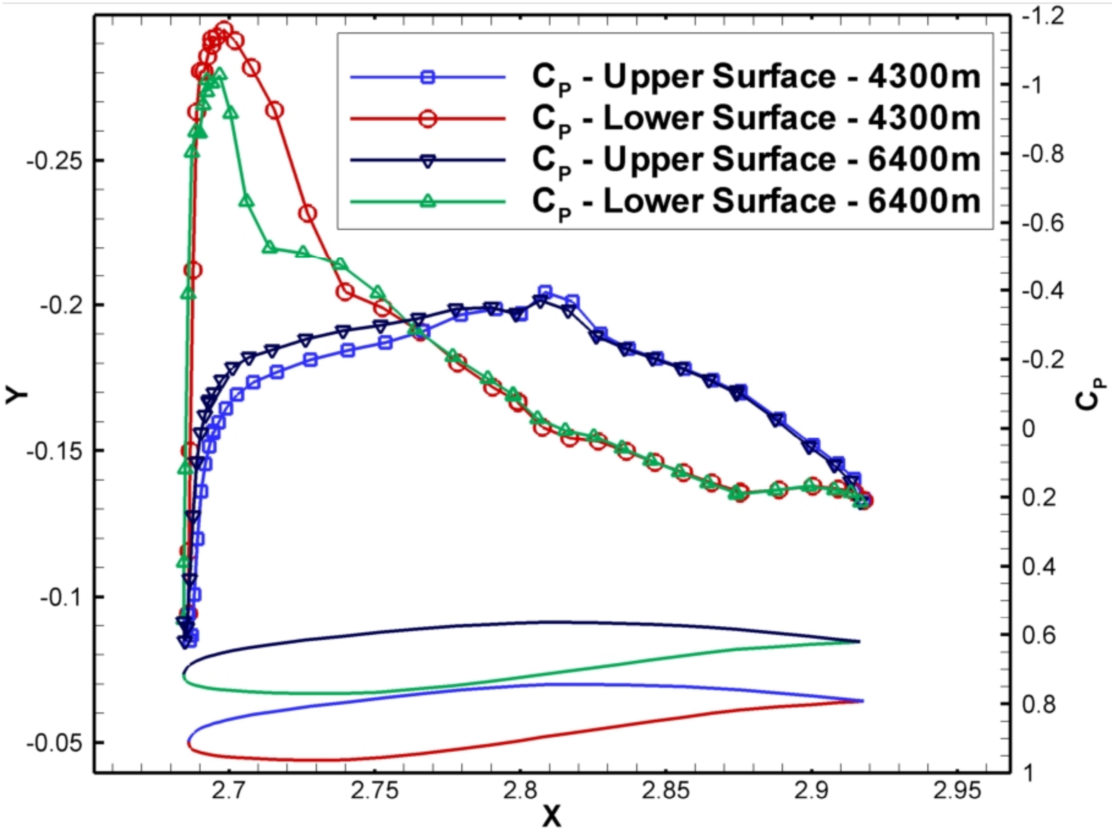
0.04
0.03
0.02
0.01
0
-0.01
-0.02
-0.03
-0.04
-0.05
ΔFreq - Hz
2400
2500
2600
2700
2800
2900
3000
ρ
Mode 1
Mode 2
Mode 4

MDO Wing



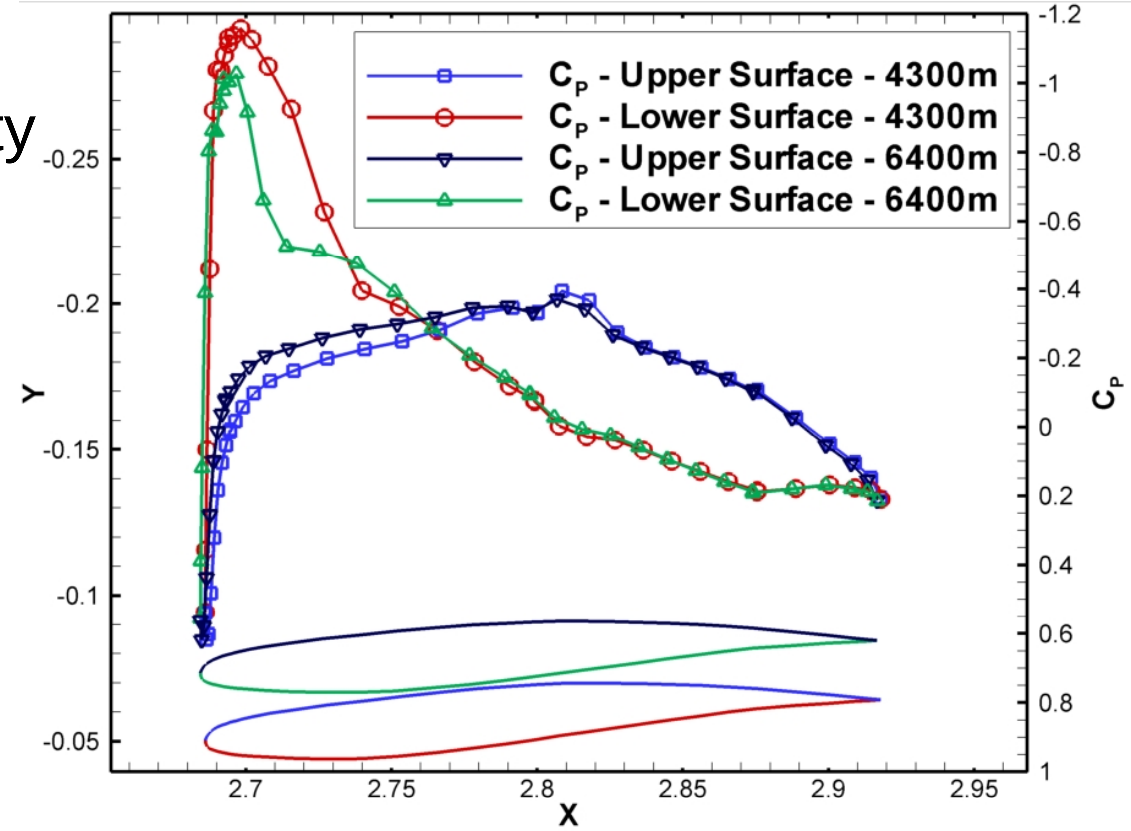
The frequency variation is consistent with the sensitivity analysis results

MDO Wing – Aerostatic effects



MDO Wing – Aerostatic effects

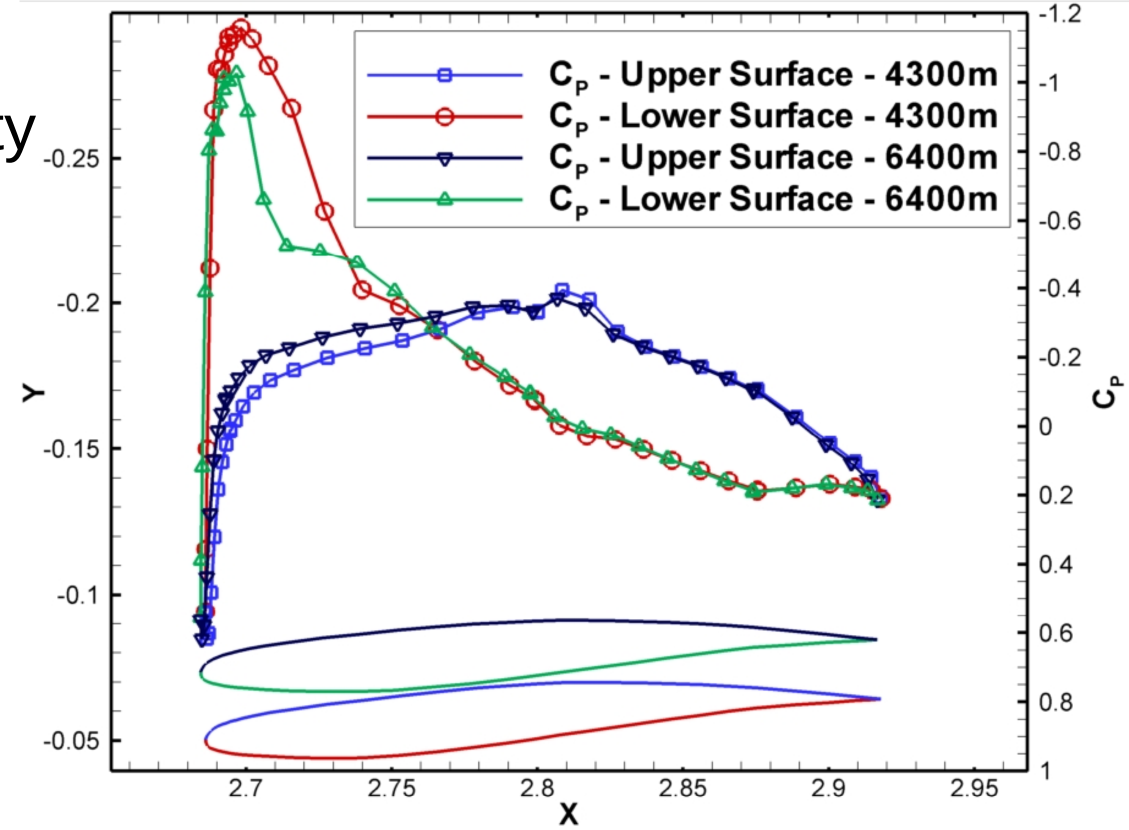
- Route 3 shows larger effect from material density



MDO Wing – Aerostatic effects

- Route 3 shows larger effect from material density

- Heavier wing reduces static deflection, reducing the effective aoa (due to twist)



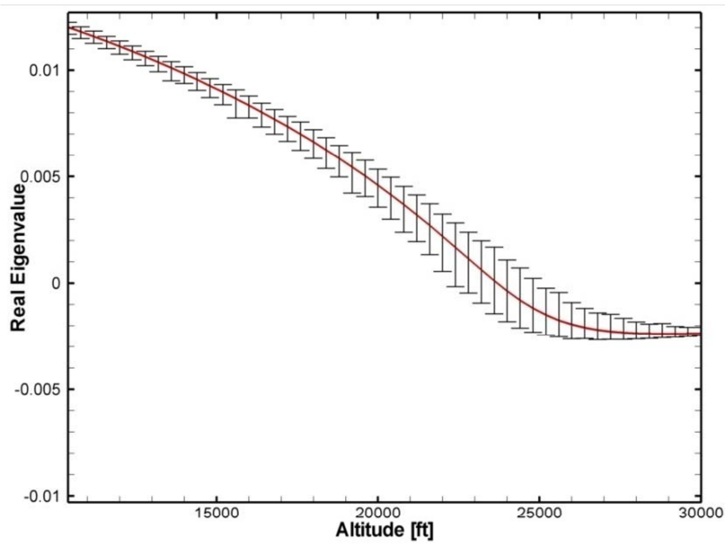
CONCLUSION

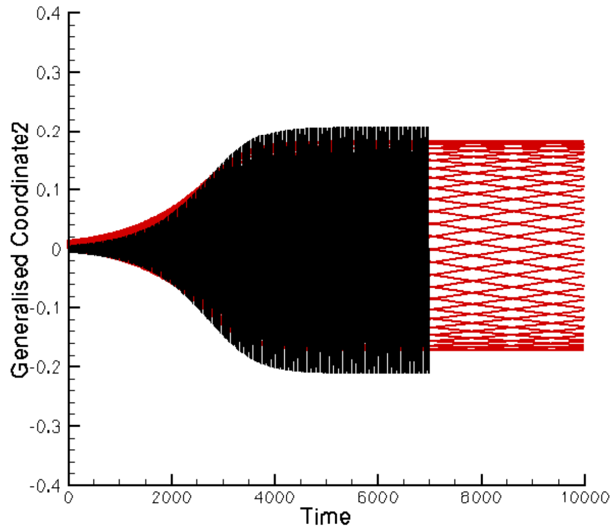
- The viability of a fast method for aeroelastic analysis has been demonstrated and applied to flutter uncertainty analysis
- A systematic approach is made possible by the formulation that allows three possibles routes of influence to be isolated.
- Three test cases have been analysed, including a realistic transport wing and a generic fighter aircraft.
- Goland wing test case shows the frequency variation alone (route 1) can be misleading and mode shape variablity is required.
- Aeroestatic effects (route 3) can have a significant influence on the variability of the instability boundary.

Future Work

- Atmospheric Variability

- Uncertainty quantification of LCO via model reduction





Eigenvalue variability due to Mach n. variability using PCE

Reduced: 2 degrees of freedom

Full: 125 thousand degrees of freedom

Q & A

Thank you for your Attention.

SCHUR METHOD

- The new formulation is solved by Newton´s Method

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\begin{bmatrix} S(\lambda) - \lambda I & \dfrac{\partial S(\lambda)}{\partial \lambda}\mathbf{p}_s - \mathbf{p}_s \\ \mathbf{q} & 0 \end{bmatrix}$$

$$\mathbf{E} = S(\lambda)\mathbf{p}_s - \lambda \mathbf{p}_s$$

$$S(\lambda) = A_{ss} - A_{sf}(A_{ff} - \lambda I)^{-1} A_{fs}$$

$$\mathbf{u} = [\mathbf{p}_s \ \lambda]^T$$

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Full Evaluation, Expensive

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$$\mathbf{u} = [\mathbf{p}_s \ \lambda]^T$$

Full Evaluation, Expensive

But $(A_{ff} - \lambda I)^{-1} \approx A_{ff}^{-1} + \lambda A_{ff}^{-1} A_{ff}^{-1} + \lambda^2 A_{ff}^{-1} A_{ff}^{-1} A_{ff}^{-1} + \ldots$

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Pre-Compute $A_{sf} A_{ff}^{-1} A_{fs}$ and $A_{sf} A_{ff}^{-2} A_{fs}$

Badcock and Woodgate, AIAA J. 48(6), 2010

SCHUR METHOD

- The new formulation is solved by Newton´s Method

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$$\mathbf{u} = \left[\mathbf{p}_s \;\; \lambda\right]^T$$

Full Evaluation, Expensive

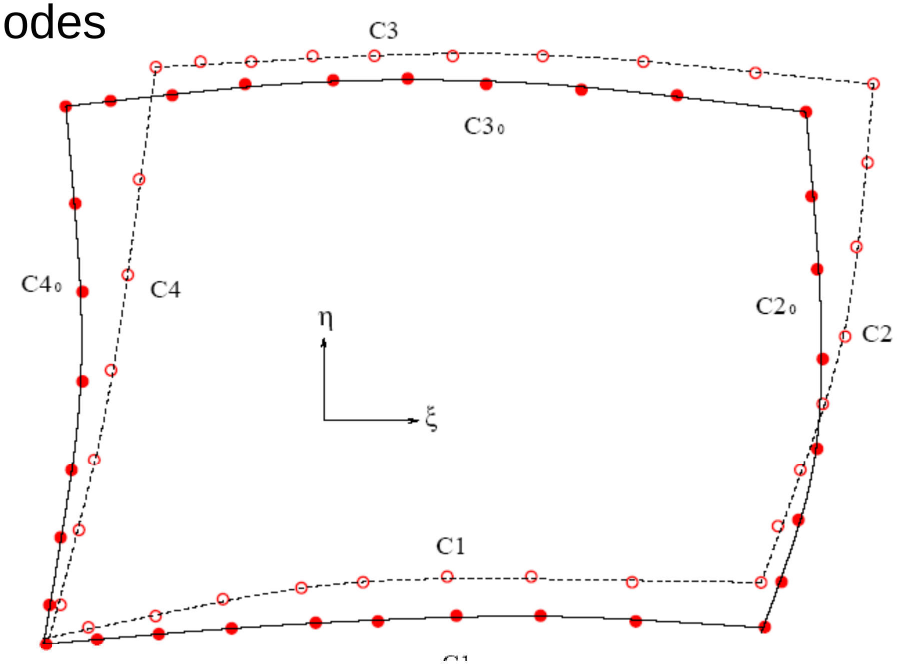
But $$(A_{ff} - \lambda I)^{-1} \approx A_{ff}^{-1} + \lambda A_{ff}^{-1} A_{ff}^{-1} + \lambda^2 A_{ff}^{-1} A_{ff}^{-1} A_{ff}^{-1} + \ldots$$

Pre-Compute $$A_{sf} A_{ff}^{-1} A_{fs} \text{ and } A_{sf} A_{ff}^{-2} A_{fs}$$

Badcock and Woodgate, AIAA J. 48(6), 2010

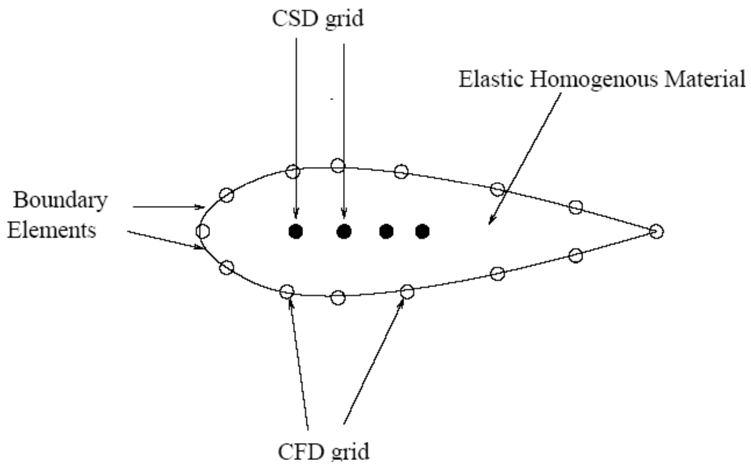
MESH MOTION

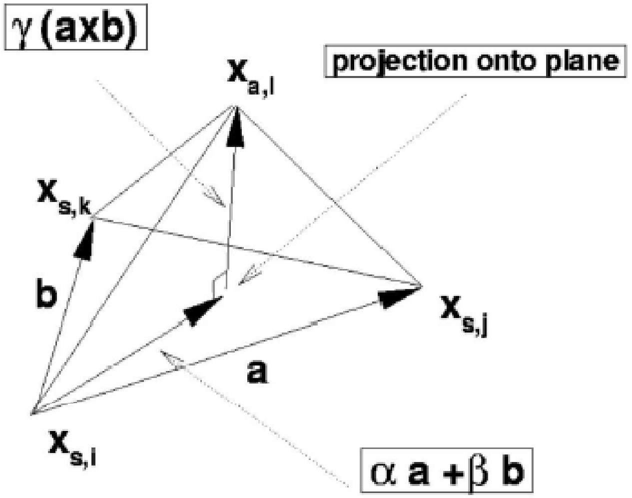
- Grid Deformations are computed by a TFI method
  - Applied only in blocks where there are surface deformations
  - TFI algorithm has 3 steps:
    - Apply linear Interpolation to block nodes to new positions
    - Apply TFI to face nodes
    - Apply TFI to interior nodes



MESH COUPLING

- Forces are tranfered between grids through CVT method
  - The structural grid is discretised with triangles
  - Each fluid point is associated with a structural element





Rampurawala, A.M. and Badcock. K.J., Evaluation of a Simplified Grid Treatment for Oscillating Trailing-Edge Control Surfaces, Journal of Aircraft, 44(4), 1177-1188, 2007.
Goura, G., Badcock, K. and Woodgate, M., Extrapolation Effects on Coupled Computational Fluid Dynamics/Computational Structural Dynamics Simulations, AIAA Journal, 41(2), 2003, 312-314.

SCHUR METHOD

- The new formulation is solved by Newton´s Method

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E}(\mathbf{w}_0, \lambda, \mathbf{p}_s, \phi, \omega) = \left[(A_{ss} - \lambda) - A_{sf} (A_{ff} - \lambda I)^{-1} A_{fs} \right] \mathbf{p}_s - \lambda \mathbf{p}_s$$

Full Evaluation, Expensive

SCHUR METHOD

- The new formulation is solved by Newton´s Method

$$\frac{\partial \mathbf{E}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E}(\mathbf{w}_0, \lambda, \mathbf{p}_s, \phi, \omega) = \left[(A_{ss} - \lambda) - A_{sf} (A_{ff} - \lambda I)^{-1} A_{fs} \right] \mathbf{p}_s - \lambda \mathbf{p}_s$$

Full Evaluation, Expensive

But $(A_{ff} - \lambda I)^{-1} \approx A_{ff}^{-1} + \lambda A_{ff}^{-1} A_{ff}^{-1} + \lambda^2 A_{ff}^{-1} A_{ff}^{-1} A_{ff}^{-1} + \ldots$

Pre-Compute $A_{sf} A_{ff}^{-1} A_{fs}$ and $A_{sf} A_{ff}^{-2} A_{fs}$

Schur complement version

(Bekas and Saad, SIAM Journal of Scientific Computing 27(2) 458, 2005)

$$S(\lambda)p_s = \lambda p_s$$

$$S(\lambda) = \left(A_{ss} - \lambda I\right) - A_{sf}\left(A_{ff} - \lambda I\right)^{-1} A_{fs}$$

Evaluate using pre-computed series approximation ← $\frac{\partial \mathbf{F}}{\partial \mathbf{u}}$

$$\frac{\partial \mathbf{F}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{F}$$

$\mathbf{F}$ →

2 options:

"full" – 1 linear solve against $A_{fs}\mathbf{p}_s$

"series" – pre-computed series approximation

SCHUR METHOD

- Series Expansion and Mode shift

$$S(\lambda) = \left(A_{ss} - \lambda I\right) - A_{sf}\left(A_{ff} - \lambda I\right)^{-1} A_{fs}$$

$$(A_{ff} - \lambda I)^{-1} \approx A_{ff}^{-1} + \lambda A_{ff}^{-1} A_{ff}^{-1} + \lambda^2 A_{ff}^{-1} A_{ff}^{-1} A_{ff}^{-1} + \dots$$

$\rightarrow \lambda$ needs to be small for series to converge

- Choose normal mode to track and apply shift

$$S(\lambda) = (A_{ss} - \lambda I - \lambda_0 I) - A_{sf}(A_{ff} - \lambda I - \lambda_0 I)^{-1} A_{fs}$$

$$A_{sf}(A_{ff} - \lambda_0 I)^{-1} A_{fs} \qquad A_{sf}(A_{ff} - \lambda_0 I)^{-2} A_{fs}$$

VARIABILITY ROUTES

- Route 1

$$\frac{\partial \overline{\mathbf{E}}_{\boldsymbol{\sigma}}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}_{\boldsymbol{\sigma}}$$

$$\mathbf{E}_{\boldsymbol{\sigma}} = \mathbf{E}_{\boldsymbol{\sigma}}\left(\overline{\mathbf{w}_{\mathbf{0}}}, \lambda, \mathbf{p}_{s}, \bar{\phi}, \omega_{j}\right)$$

VARIABILITY ROUTES

- Route 1

$$\frac{\partial \overline{\mathbf{E}}_{\boldsymbol{\sigma}}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}_{\boldsymbol{\sigma}}$$

$$\mathbf{E}_{\boldsymbol{\sigma}} = \mathbf{E}_{\boldsymbol{\sigma}}\left(\overline{\mathbf{w}_{0}, \lambda, \mathbf{p}_{s}}, \bar{\phi}, \omega_{j}\right)$$

• Computational cost is reduced to pre-computing matrices and 1 steady state calculation
• E depends on the mean case, except for ω

VARIABILITY ROUTES

- Route 2

$$\frac{\partial \overline{\mathbf{E}}_{\boldsymbol{\sigma}}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E} = \mathbf{E}\left(\overline{\mathbf{w}_{\mathbf{0}}}, \lambda, \mathbf{p}_s, \phi_j, \omega_j\right)$$

VARIABILITY ROUTES

- Route 3

$$\frac{\partial \mathbf{E}_{\boldsymbol{\sigma}}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

$$\mathbf{E} = \mathbf{E}\left(\mathbf{w}_{\mathbf{0}j}, \lambda, \mathbf{p}_s, \phi_j, \omega_j\right)$$

VARIABILITY ROUTES

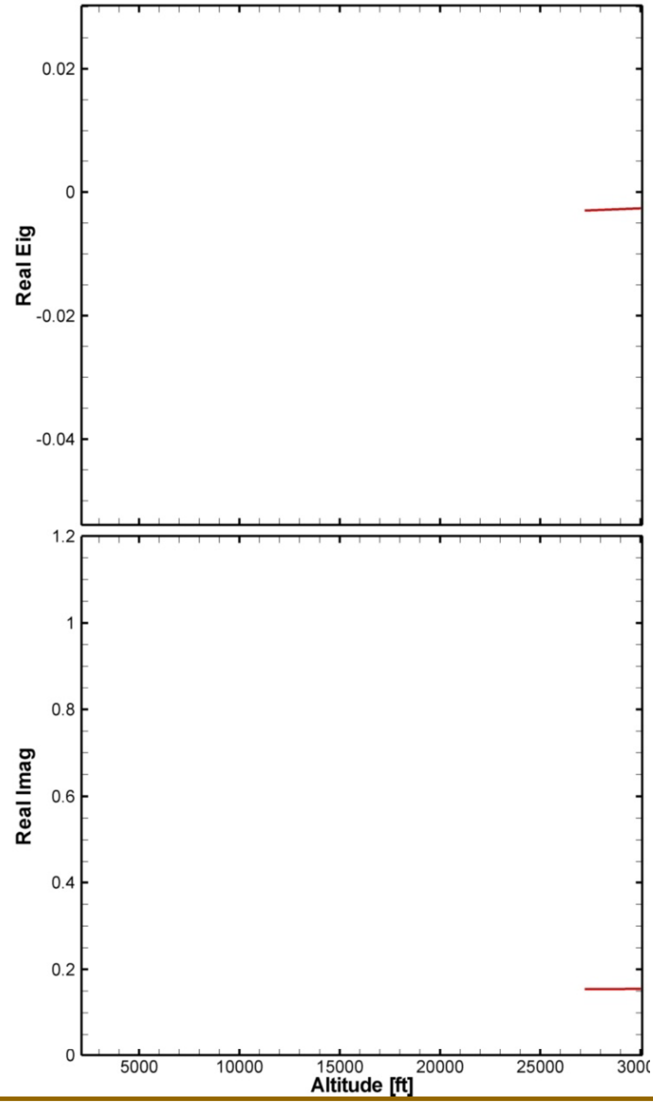
- Route 3

$$\frac{\partial \mathbf{E}_{\boldsymbol{\sigma}}}{\partial \mathbf{u}} \Delta \mathbf{u} = -\mathbf{E}$$

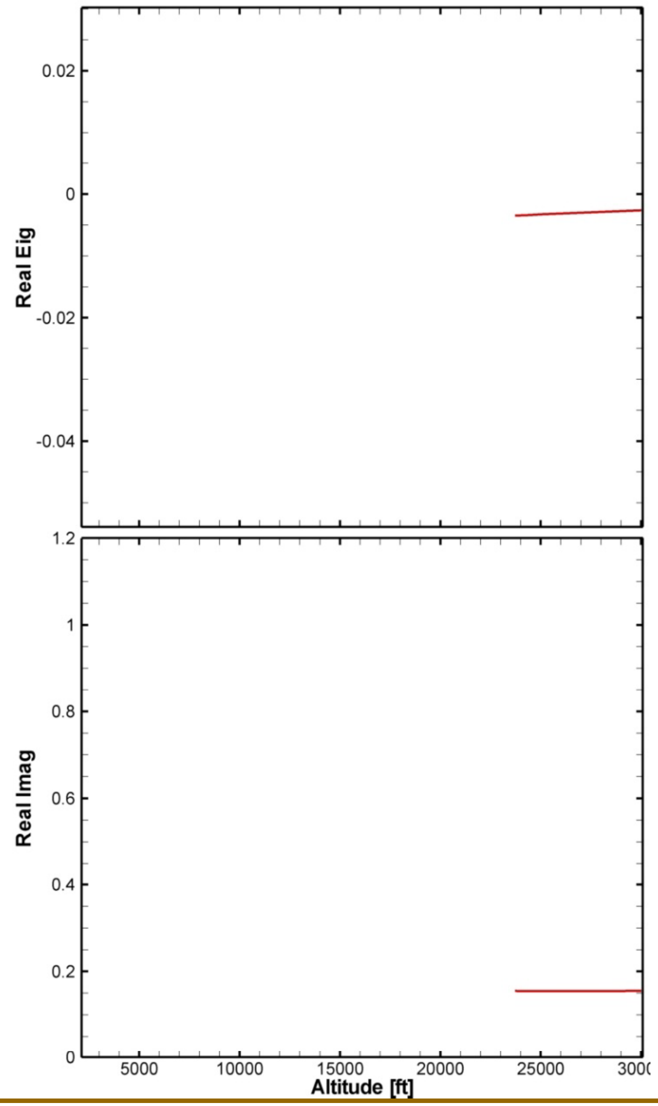
$$\mathbf{E} = \mathbf{E}\left(\mathbf{w}_{\mathbf{0}\,j}, \lambda, \mathbf{p}_{s}, \phi_{j}, \omega_{j}\right)$$

- **Requires computing steady state and matrices at each full evaluation step**
- **All sources of variability, for each mode set are used**

TYPICAL CALCULATION



TYPICAL CALCULATION



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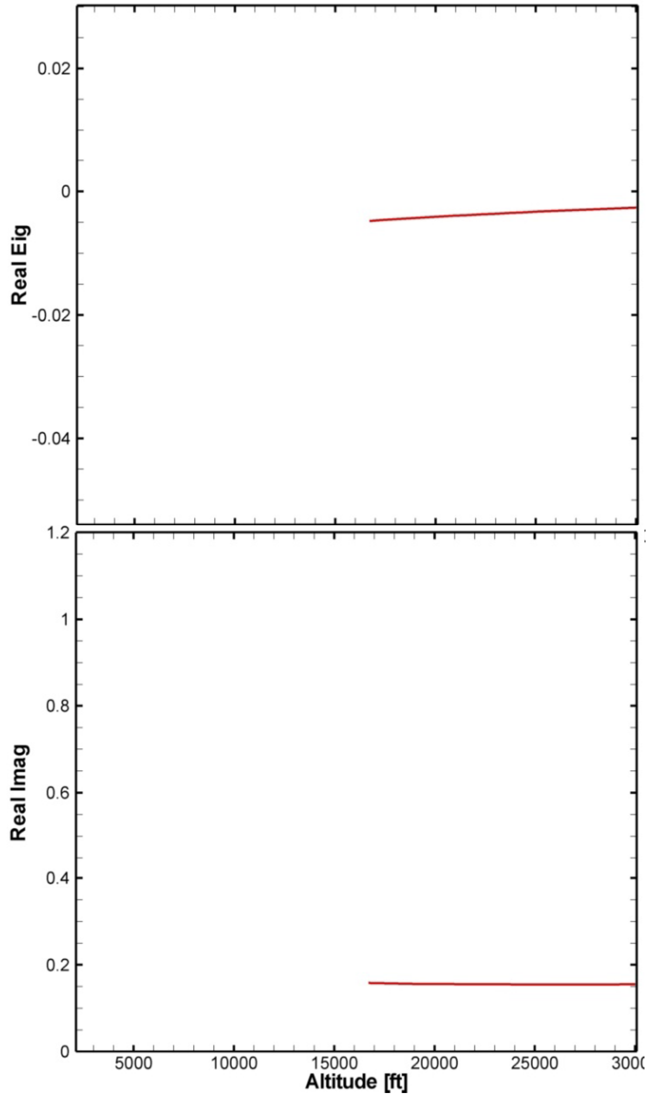

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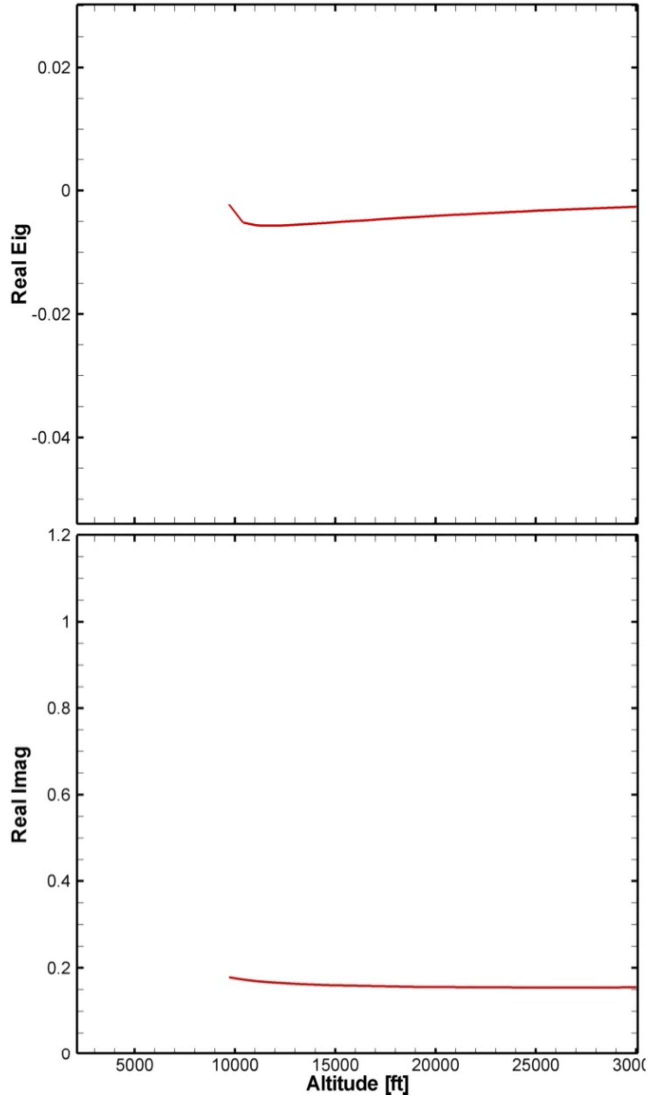

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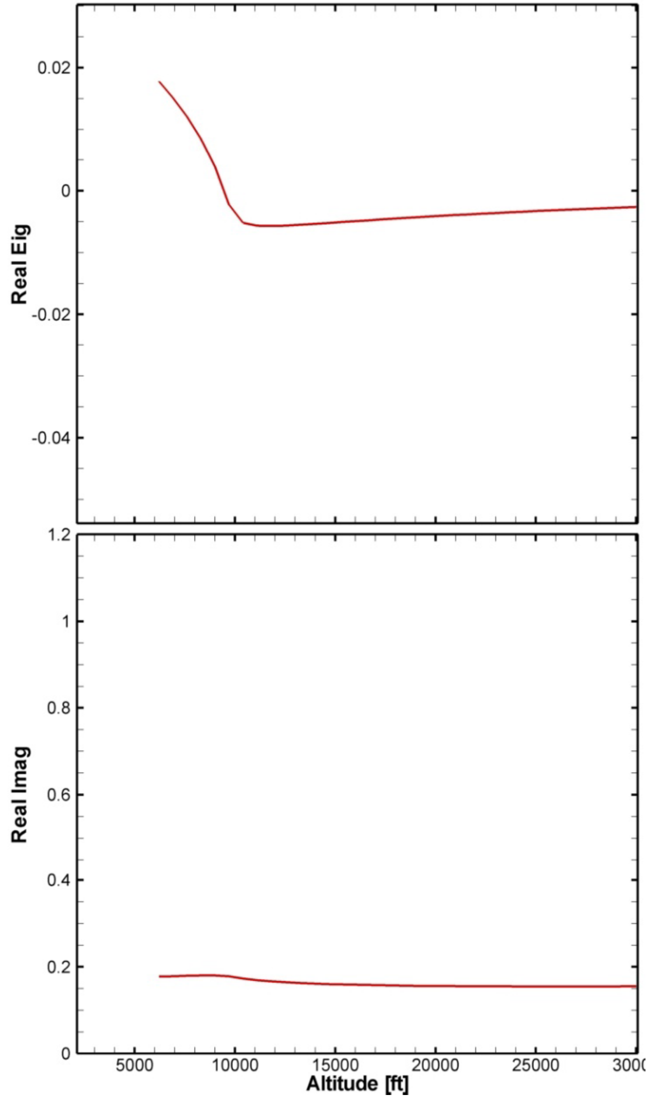

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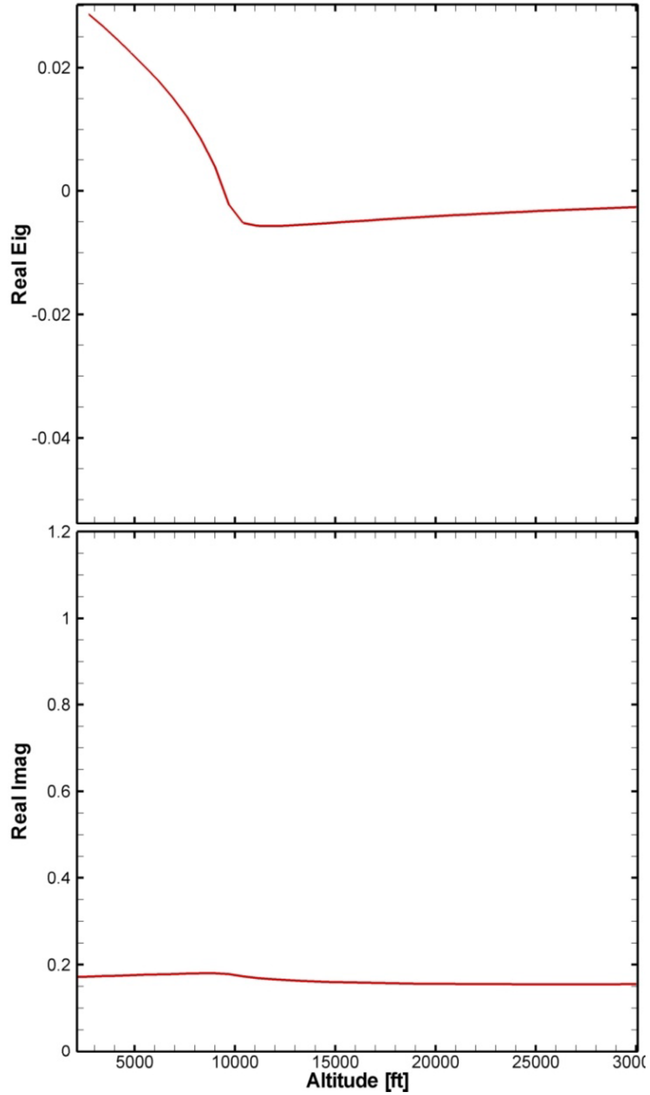

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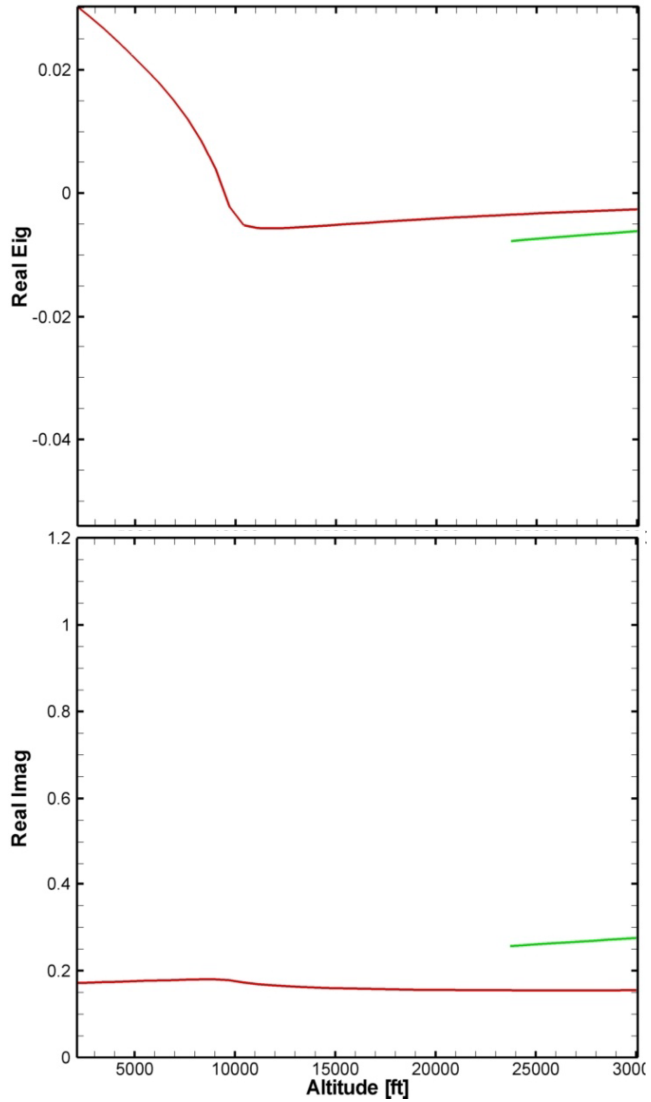

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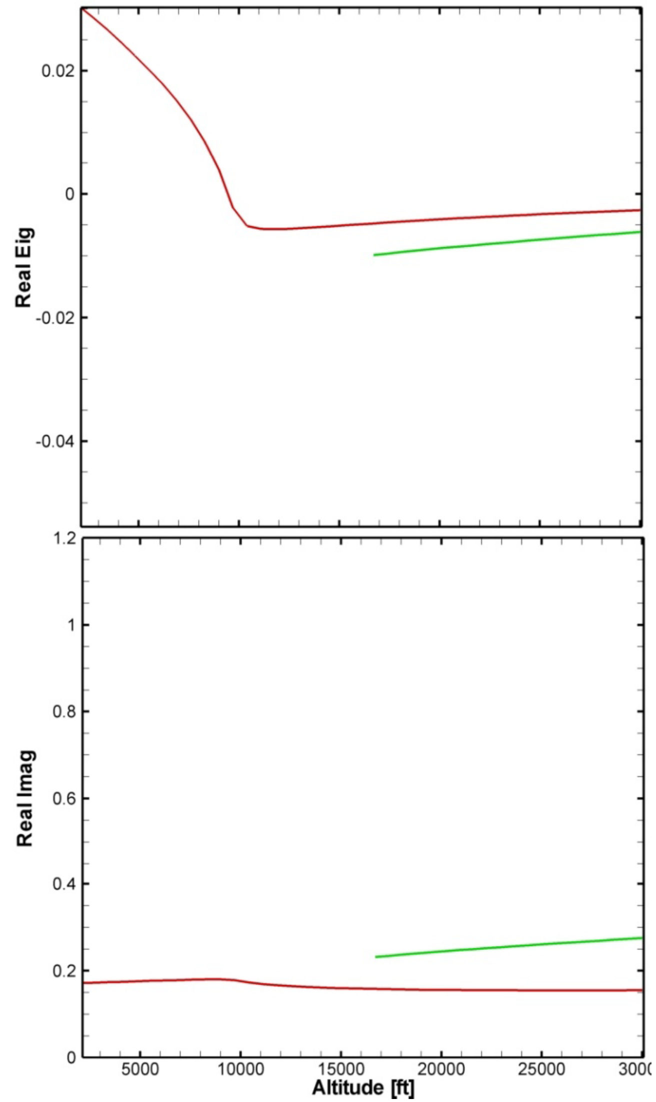

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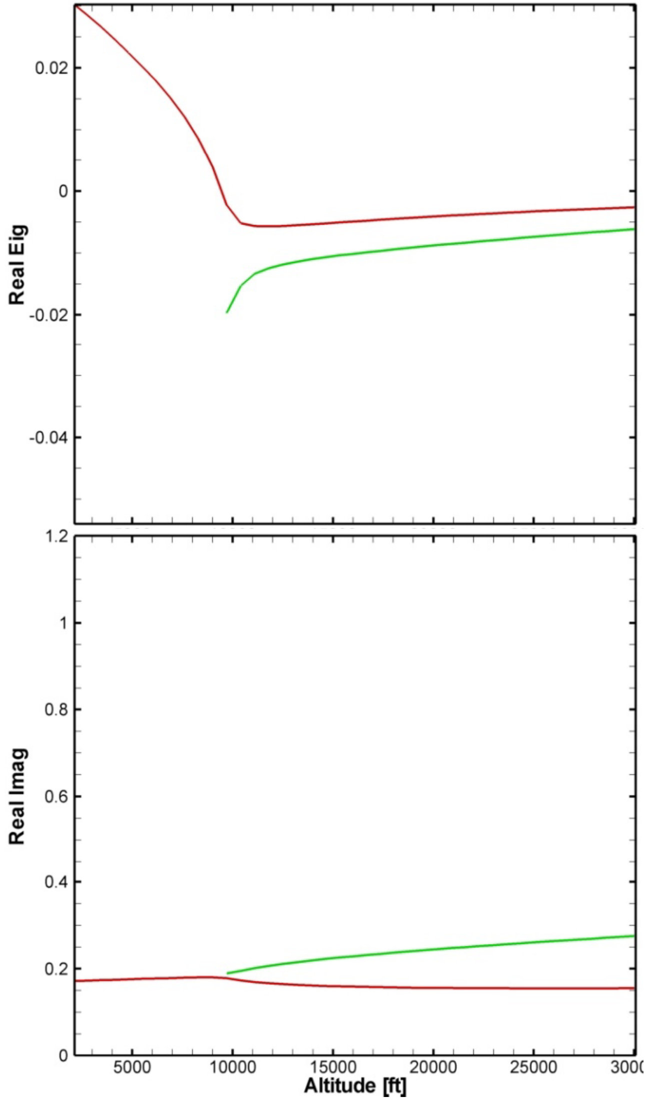

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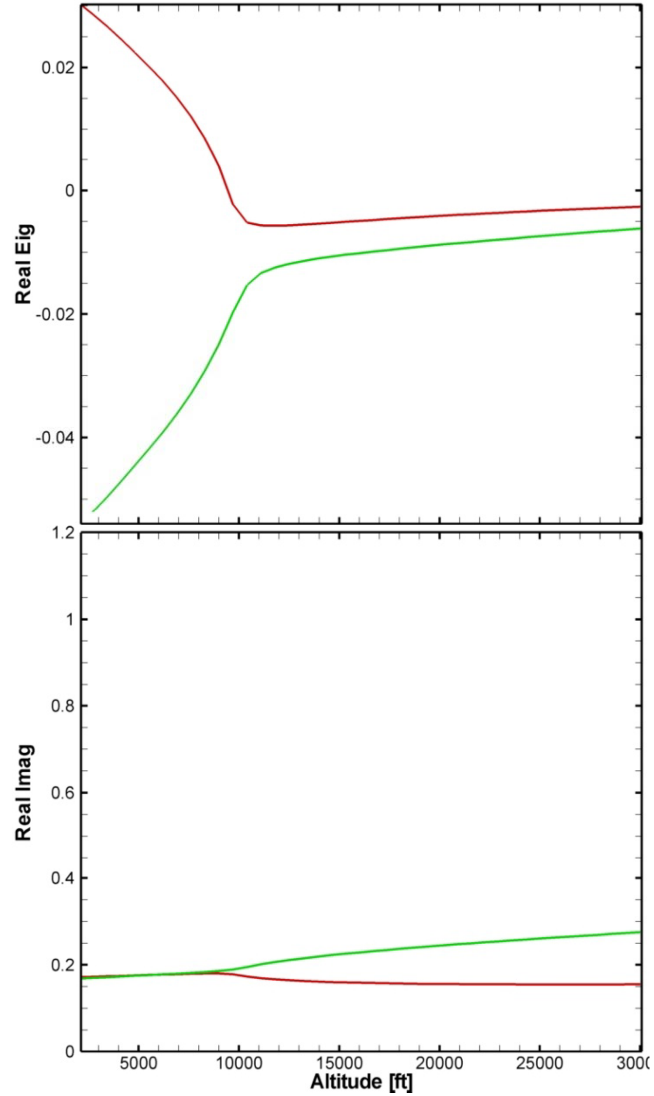

TYPICAL CALCULATION