The quantum mechanics approach to uncertainty modeling in structural dynamics

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Outline

- Introduction
  - Notions of uncertainty and probability
  - Uncertainty in structural dynamics
- Theoretical Development
  - Quantum mechanics motivation
  - Density matrix
  - Model of uncertain structures
  - Updating of means and covariances
- Example
  - Undamped three degree of freedom system

Introduction

- **Uncertainty**: State of mind of an observer of an experiment whose outcome is an event out of many possible alternatives
- **Probability Theory**: Models experiments of the previous kind and facilitates an objective measure of uncertainty: that of entropy
- **Jaynes Interpretation**:
  - Probability distribution is the knowledge of the observer about the experiment
  - Entropy is measure of ignorance

Introduction

- **Uncertainty in Structural Dynamics**
  - **Aleatoric** Lack of knowledge about the exact values of the inertia, damping and elasticity
  - **Epistemic** inability to model all the intricacies of a complicated structure
- **Density Matrix to model uncertainty**:
  - Captures both the underlying system dynamics and statistics of uncertain structures
  - Covariance properties easy to obtain
  - Facilitates uncertain model updating

Quantum Mechanics Motivation

Motivating Quantum Mechanics Principles:

- State of a quantum mechanical particle is expressed as wavefunction $\mathbf{\Phi}\left(x\right)$
- $\mathbf{\Phi}\left(x\right)^2$ is interpreted as probability density function
- Measurable physical quantities are described by Hermitian linear operators $\mathbf{O}$
- Once the system in state $\mathbf{\Phi}$ then the average value of the observed quantity corresponds to

$$\langle O \rangle = \mathbf{\Phi}^H \mathbf{O} \mathbf{\Phi}$$

Quantum Mechanics Motivation

Orthogonality relations for n-degree of freedom system described by $n \times n$ symmetric $\mathbf{M}$, $\mathbf{K}$ and $\mathbf{C}$:

$$2\omega_r\zeta_r = \frac{c_r}{m_r} = \frac{\mathbf{\Psi}_r^H \mathbf{C} \mathbf{\Psi}_r}{\mathbf{\Psi}_r^H \mathbf{M} \mathbf{\Psi}_r} \quad r = 1..n$$

$$\omega_r^2 = \frac{k_r}{m_r} = \frac{\mathbf{\Psi}_r^H \mathbf{K} \mathbf{\Psi}_r}{\mathbf{\Psi}_r^H \mathbf{M} \mathbf{\Psi}_r} \quad r = 1..n$$

	Damped	**Undamped**
Observables	$\mathbf{M}$, $\mathbf{C}$,$\mathbf{K}$	$\mathbf{M}$, $\mathbf{K}$
Observed	ω_r , $2\omega_r\zeta_r$	ω_r^2

Density Matrix

Consider a damped n-degree of freedom system:

- Normalized-to-unity complex mode shape $\boldsymbol{\Psi}_r \in C^n$

$$\boldsymbol{\Psi}_r = \sum_i^n \alpha_i \mathbf{u}_i$$

where $\{\mathbf{u}_i\}, i = 1..n$ orthonormal basis of C^n

$$\sum_i^n |\alpha_i|^2 = 1$$

- Substitute $\boldsymbol{\Psi}_r$ in the orthogonality relations

Density Matrix

$$2\omega_r\zeta_r = \frac{c_r}{m_r} = \frac{\sum_{i,j}^{n} \alpha_i^* \alpha_j C_{ij}}{\sum_{i,j}^{n} \alpha_i^* \alpha_j M_{ij}}$$

$$\omega_r^2 = \frac{k_r}{m_r} = \frac{\sum_{i,j}^{n} \alpha_i^* \alpha_j K_{ij}}{\sum_{i,j}^{n} \alpha_i^* \alpha_j M_{ij}}$$

- m_r, c_r and k_r the weighted averages of the elements of the mass, damping and stiffness matrices

Density Matrix

Definition: The $n \times n$ matrix $\mathbf{P}_r$ that the coefficients $\alpha_i^* \alpha_j$ create,

$$\mathbf{P}_r = \mathbf{\Psi}_r \mathbf{\Psi}_r^H$$

since $\mathbf{\Psi}_r$ is normalized to unity,

$$Tr\left(\mathbf{P}_r\right) = 1$$

- Orthogonality relationships can be written as,

$$2\omega_r \zeta_r = \frac{Tr(\mathbf{P}_r \mathbf{C})}{Tr(\mathbf{P}_r \mathbf{M})} \quad \omega_r^2 = \frac{Tr(\mathbf{P}_r \mathbf{K})}{Tr(\mathbf{P}_r \mathbf{M})}$$

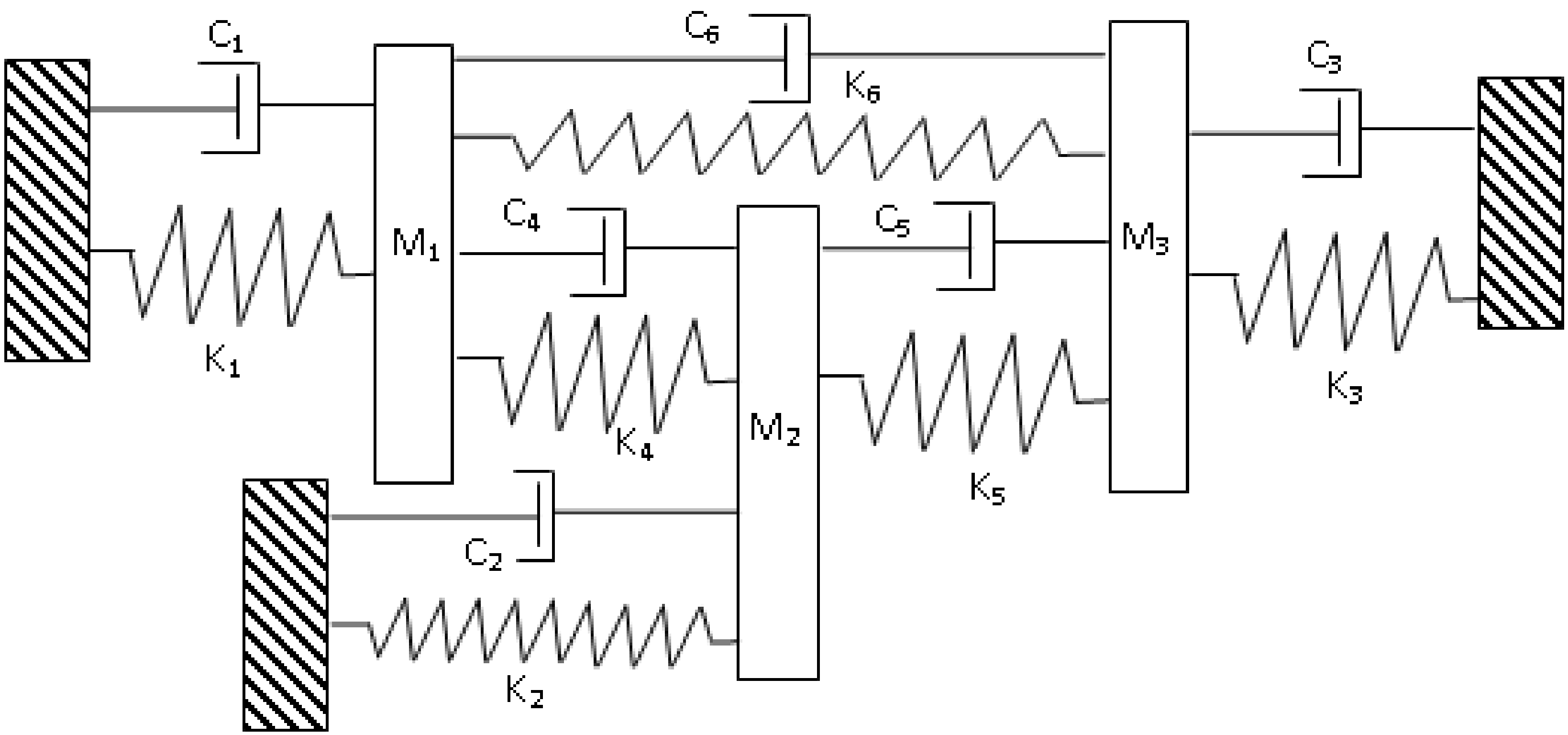
Uncertain Structures: Modeling

How are uncertain structures modeled?

- Model Nominal Dynamics
  - Nominal mass, $\mathbf{M}$, stiffness, $\mathbf{K}$ and damping $\mathbf{C}$ matrices
- Model for Generating Statistics
  - Average (nominal) density matrix for each mode
- Set of density matrices $\{\mathbf{P}_r\ 1 \leq r \leq n\}$ together with $\mathbf{M}$, stiffness, $\mathbf{K}$ and damping $\mathbf{C}$ matrices constitute the uncertain structure

Example

Three Degree of Freedom System



Undamped System: Non-random mass

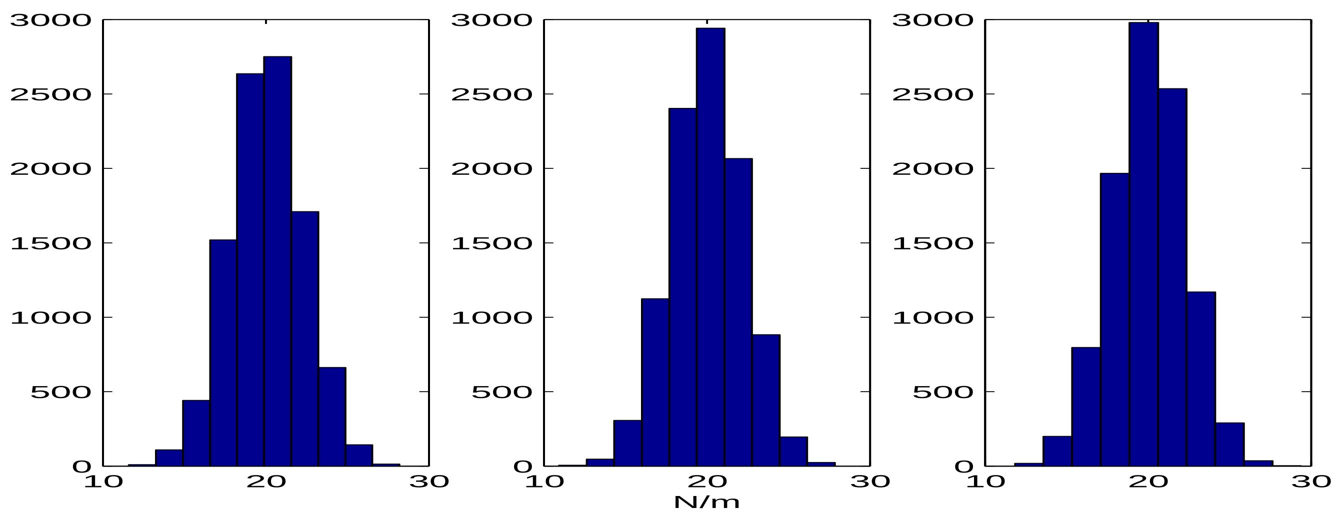
Parameter Values of Nominal System

$$M_1 = 2, M_2 = 3, M_3 = 4Kg$$

$$K_3 = 10, K_4 = 10, K_6 = 30, K_1 = K_2 = K_5 = 20\frac{N}{m}$$

Uncertain System

$$K_i = \left(20 + \sqrt{5}\mathcal{N}\left(0,1\right)\right)\frac{N}{m} \quad i = 1, 2, 5$$



Undamped System: Non-random mass

- Sample: 10000 realizations
- Average density matrices $\bar{\mathbf{P}}_r$ $1 \leq r \leq 3$ were computed
  - Observer's state of knowledge about the uncertain system

$$\omega_r^2 = \frac{Tr(\mathbf{P}_r \mathbf{K})}{Tr(\mathbf{P}_r \mathbf{M})}, \left(\frac{rad}{s}\right)^2 \qquad \omega = \frac{Tr\left(\mathbf{P}_r \sqrt{\mathbf{K}}\right)}{Tr\left(\mathbf{P}_r \sqrt{\mathbf{M}}\right)}, \frac{rad}{s}$$

$$\begin{pmatrix} 5.2284\,(5.2082) \\ 20.9078\,(20.9241) \\ 35.5304\,(35.5912) \end{pmatrix} \qquad \begin{pmatrix} 2.2964\,(2.2813) \\ 4.5723\,(4.5719) \\ 5.9354\,(5.9654) \end{pmatrix}$$

Undamped System: Non-random mass

Analysis of Covariance

- Take Cholesky decomposition of mass matrix

$$\mathbf{M} = \mathbf{R}^T\mathbf{R}$$

- Substitute in orthogonality relation to get

$$\omega_r^2 = \frac{\Psi^T\mathbf{K}\Psi}{\Psi^T\mathbf{R}^T\mathbf{R}\Psi}$$

- This is equivalent to

$$\omega_r^2 = \frac{\Psi^T\mathbf{R}^T\mathbf{R}^{-T}\mathbf{K}\mathbf{R}^{-1}\mathbf{R}\Psi}{\Psi^T\mathbf{R}^T\mathbf{R}\Psi}$$

Undamped System: Non-random mass

Analysis of Covariance

- Matrix $\mathbf{A} = \mathbf{R}^{-T}\mathbf{K}\mathbf{R}^{-1}$ is symmetric
- Same eigenvalues as of the original system but the associated eigenvectors are given by $\mathbf{R}\Psi$
- New density matrices from the previous ones

$$\mathbf{P}_i^{\mathbf{A}} = \frac{1}{Tr\left(\mathbf{R}\mathbf{P}_i\mathbf{R}^T\right)}\left(\mathbf{R}\mathbf{P}_i\mathbf{R}^T\right)$$

Uncertain System: Non-random mass

Analysis of Covariance

- **Hypothesis**: Variability in the sample of natural frequencies is due to system covariance matrices
  - $n^2 \times n^2$ covariance matrices of vectorized $\mathbf{A}$
- Extending the theory of density matrices to tensor product $R^n \otimes R^n$, new density matrices are defined,

$$\mathbf{P}_{ij}^{\mathbf{A}} = \mathbf{P}_i^{\mathbf{A}} \otimes \mathbf{P}_j^{\mathbf{A}}, \;\; 1 \leq i, j \leq n$$

Uncertain System: Non-random mass

Analysis of Covariance

$$Cov\left(\omega_i^2, \omega_j^2\right) = Tr\left(\bar{\mathbf{P}}_{ij}^{\mathbf{A}} Cov\left(vec\left(\mathbf{A}\right)\right)\right)$$

- Since the mass matrix is non-random $Cov\left(\omega_i^2, \omega_j^2\right)$ depends on the $Cov\left(vec\left(\mathbf{K}\right)\right)$
- $Cov\left(vec\left(\mathbf{K}\right)\right)$ is easily constructed from the variances and covariances of the individual stiffness elements

Undamped System: Non-random mass

Analysis of Covariance

$$vec\mathbf{K} = \begin{bmatrix} K_1 + K_4 + K_6 \\ -K_4 \\ -K_6 \\ -K_4 \\ K_2 + K_4 + K_5 \\ -K_5 \\ -K_6 \\ -K_5 \\ K_3 + K_5 + K_6 \end{bmatrix}$$

Undamped System: Non-random mass

Analysis of Covariance

- To construct the 9×9, $Cov\left(vec\mathbf{K}\right)$, use

$$\mathbf{Var}\left(K_i + K_j + K_k\right) =$$
$$Var\left(K_i\right) + Var\left(K_j\right) + Var\left(K_k\right) +$$
$$2\left(Cov\left(K_i, K_j\right) + Cov\left(K_i, K_k\right) + Cov\left(K_j, K_k\right)\right)$$
$$\mathbf{Cov}\left(K_i + K_j + K_k, -K_k\right) =$$
$$-Var\left(K_k\right) - Cov\left(K_i, K_k\right) - Cov\left(K_j, K_k\right)$$
$$\mathbf{Cov}\left(K_i + K_j + K_k, K_i + K_l + K_m\right) =$$
$$Var\left(K_i\right) + Cov\left(K_i, K_l\right) + Cov\left(K_i, K_m\right)$$
$$+Cov\left(K_j, K_m\right) + Cov\left(K_k, K_i\right) + Cov\left(K_k, K_l\right) +$$
$$Cov\left(K_k, K_m\right)$$

Undamped System: Non-random mass

Analysis of Covariance

- Since $\mathbf{A} = \mathbf{R}^{-T}\mathbf{K}\mathbf{R}^{-1}$ then

$$vec\,(\mathbf{A}) = \left(\mathbf{R}^{-T} \otimes \mathbf{R}^{-T}\right) vec\,(\mathbf{K})$$

- Covariance of $\mathbf{A}$ is given by

$$Cov\,(vec\,(\mathbf{A})) =$$

$$\left(\mathbf{R}^{-T} \otimes \mathbf{R}^{-T}\right)^{T} Cov\,(\mathbf{K}) \left(\mathbf{R}^{-T} \otimes \mathbf{R}^{-T}\right)$$

Undamped System: Non-random mass

Analysis of Variance: Results, $Cov\left(\omega_i^2, \omega_j^2\right)$

Using covariance expressions for $Cov\left(vec\left(\mathbf{K}\right)\right)$

$$\begin{pmatrix} 0.0758\,(0.0767) & 0.1327\,(0.1325) & 0.1663\,(0.1677) \\ 0.1327\,(0.1325) & 1.7547\,(1.7709) & 0.1385\,(0.1349) \\ 0.1663\,(0.1677) & 0.1385\,(0.1349) & 0.8012\,(0.8023) \end{pmatrix}$$

Using sample covariance for $Cov\left(vec\left(\mathbf{K}\right)\right)$

$$\begin{pmatrix} 0.0764\,(0.0767) & 0.1323\,(0.1325) & 0.1667\,(0.1677) \\ 0.1323\,(0.1325) & 1.7702\,(1.7709) & 0.1404\,(0.1349) \\ 0.1667\,(0.1677) & 0.1404\,(0.1349) & 0.8019\,(0.8023) \end{pmatrix}$$

Undamped system: Non-random mass

Model Updating: Mean

- **Problem formulation:** $\mathbf{K}_{nom}$ is unknown and an initial guess $\mathbf{K}_{in}$ is given. Density matrices and nominal natural frequencies ω_i^2 are known.

$$\mathbf{K}_{in} = \mathbf{K}_{nom} + \delta\mathbf{K}$$

- **Objective**: Find $\delta\mathbf{K}$

- **Solution**

$$\omega_i^2 = \frac{Tr\left(\rho_i \mathbf{K}_{nom}\right)}{Tr\left(\rho_i \mathbf{M}\right)} = \frac{Tr\left(\rho_i \left(\mathbf{K}_{in} - \delta\mathbf{K}\right)\right)}{Tr\left(\rho_i \mathbf{M}\right)}$$

Undamped system: Non-random mass

- System of Updating Equations

$$Tr\left(\rho_i \delta \mathbf{K}\right) = Tr\left(\rho_i \mathbf{K}_{in}\right) - \omega_i^2 Tr\left(\rho_i \mathbf{M}\right) \quad 1 \leq i \leq n$$

Initial Stiffness Matrix, $\mathbf{K}_{in}$, and $\delta \mathbf{K}$

$$\begin{bmatrix} K_1 + K_4 + K_6 & -K_4 & -K_6 \\ -K_4 & K_2 + K_4 + K_5 & -K_5 \\ -K_6 & -K_5 & K_3 + K_5 + K_6 \end{bmatrix}$$

$$\begin{bmatrix} \delta K_1 & 0 & 0 \\ 0 & \delta K_2 + \delta K_5 & -\delta K_5 \\ 0 & -\delta K_5 & \delta K_5 \end{bmatrix}$$

Undamped System: Non-random mass

Numerical example

- Initial Nominal Values

$$\mathbf{K}_i = 10\frac{N}{m} \quad i = 1, 2, 5$$

- Substituting the known ω_n^2 and ρ_is in the system of updating equations

$$0.083\delta\mathbf{K_1} + 0.091\delta\mathbf{K_2} + 0.006\delta\mathbf{K_5} = -1.77$$

$$0.19\delta\mathbf{K_1} + 0.24\delta\mathbf{K_2} + 0.54\delta\mathbf{K_5} = -7.98$$

$$0.398\delta\mathbf{K_1} + 0.001\delta\mathbf{K_2} + 0.037\delta\mathbf{K_5} = -4.38$$

Undamped System: Non-random mass

- Solution

$$\begin{bmatrix} \delta\mathbf{K}_1 \\ \delta\mathbf{K}_2 \\ \delta\mathbf{K}_5 \end{bmatrix} = \begin{bmatrix} -10.11 \\ -9.64 \\ -10.5 \end{bmatrix} \frac{N}{m}$$

- Nominal and updated nominal stiffness matrices, where $\mathbf{K}_{nomUP} = \mathbf{K}_{\mathbf{in}} - \delta\mathbf{K}$

$$\begin{bmatrix} 60\,(60.12) & -10\,(-10) & -30\,(-30) \\ -10\,(-10) & 50\,(49.69) & -20\,(-20.05) \\ -30\,(-30) & -20\,(-20.05) & -60\,(-60.05) \end{bmatrix}$$

Undamped System: Non Random Mass

Model Updating: Variance

- **Problem Formulation**: Variances of $\mathbf{K}_i$ $i = 1, 2, 5$ are not known. No initial variances are required. Density matrices and variances of ω_i^2 are known.
- **Objective**: Find the variances of individual stiffness parameters
- **Solution**

$$var\left(\omega_i^2\right) = Tr\left(\bar{P}_{ii}^{A} cov\left(vecA\right)\right) \; 1 \leq i \leq n$$

Undamped System: Non Random Mass

- Built up $cov\left(vec\left(K\right)\right)$ using the individual variances and covariances
- Set

$$Cov\left(vec\left(\mathbf{A}\right)\right) =$$

$$\left(\mathbf{R}^{-T} \otimes \mathbf{R}^{-T}\right)^{T} Cov\left(\mathbf{K}\right)\left(\mathbf{R}^{-T} \otimes \mathbf{R}^{-T}\right)$$

- Substitute the observed $var\left(\omega_i^2\right)$ and $\bar{P}_{ii}^{A}$ in the last expression of the previous slide to get three equations in the three unknown variances

Undamped System: Non Random Mass

$$0.006838 var \mathbf{K}_1 + 0.008299 var \mathbf{K}_2 + 0.00003137 var \mathbf{K}_5$$

$$= 0.0767$$

$$0.00035 var \mathbf{K}_1 + 0.05813 var \mathbf{K}_2 + 0.2925 var \mathbf{K}_5$$

$$= 1.7710$$

$$0.1589 var \mathbf{K}_1 + 0.000001294 var \mathbf{K}_2 + 0.001363 var \mathbf{K}_5$$

$$= 0.8023$$

Undamped System: Non Random Mass

- **Solution**

$$\begin{bmatrix} var\mathbf{K}_1 \\ var\mathbf{K}_2 \\ var\mathbf{K}_5 \end{bmatrix} = \begin{bmatrix} 5.0360 \\ 5.0982 \\ 5.0067 \end{bmatrix}$$

Conclusions

- Uncertainty model based on the concept of density matrix
  - Reformulation of orthogonality relationships using the trace operator
  - In uncertainty context these give the estimated mean values
- Theory extended through tensor products to account for covariances
- Updating of means and computation of unknown covariances

Future Work

Results not presented

- Damped systems
- Uncertain random mass. Distribution in natural frequencies and decay rates are non-Gaussian

Future Work

- Frequency domain formulation and structural modification of uncertain structures
- The uncertain system could be treated as if it is certain