Reduced model construction for and analysis of the saline boundary layer problem

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At the subsurface of salt lakes saline boundary layers are often present. This boundary layer is a result of the evaporation induced upward throughflow at the surface of the underlying porous medium. When this upward transport is balanced by salt dispersion, a steady state boundary layer is formed. However, this boundary layer can be unstable when perturbed. This results in a complicated groundwater motion and transport of contaminants that may be disposed of in the saline lake. Thence, when considering saline lakes as evaporation basins for irrigation waste waters, the groundwater dynamics associated with convection must be well understood.

To get more insight in the magnitude and spatial structure of growing instabilities for the Wooding problem for arbitrary parameter settings, both laboratory experiments were performed and direct time-integrations of the discretized model equations were carried out. These laboratory and numerical experiments show the existence of finite-amplitude convection cells and finger-like salt distributions with a well-defined horizontal periodicity. The direct time-integrations give interesting first results concerning convective solutions, but an extensive survey of the influence of the initial conditions and parameter settings on the model results is virtually impossible.

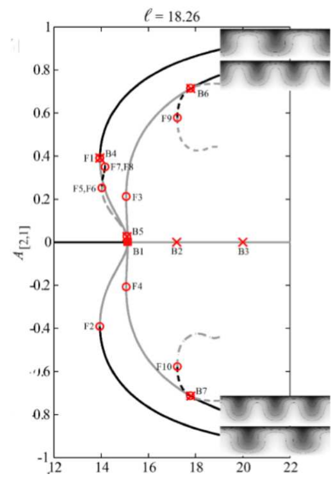

Figure 1: Example of a bifurcation diagram obtained with the reduced model. Here the amplitude of one of the Galerkin modes is shown as a function of the system Rayleigh number. The Péclet number was kept fixed.

In this contribution we discuss the numerical construction of a low–order dynamical system which can be used to systematically investigate the nonlinear behavior of the Wooding problem in a computationally inexpensive way. This is done by projecting the nonlinear model equations onto a relatively small set of eigenfunctions of the problem linearized at criticality, hence reducing the system of coupled partial differential equations to a system of ordinary differental equations. The Galerkin projection approach is complicated by the fact that the problem under consideration is non-self-adjoint due to the existing evaporation. This implies that the eigenfunctions do not form an orthogonal set and therefore the *adjoint* eigenfunctions are used for the projection. The two controlling parameters for this problem are the Rayleigh and Péclet number, which are both taken in relevant ranges. Hence the reduced model is constructed in such a way that it is capable to provide solutions in the strongly nonlinear regime as well. For the reduced model we show the number, the spatial structure, and the stability of several finite-amplitude steady-state and time-periodic solutions, as well as their particular dependence on the parameters (see for example the bifurcation diagram depicted in Figure 1). Convergence of these solutions towards the fully nonlinear model results is shown by means of direct numerical simulations. Further, the reduced model captures the main characteristics of the complex nonlinear behavior as seen in Hele–Shaw experiments by Wooding et al. [*Water Resour. Res.*, 33(6):1199–1228, 1997].