Computation and Application of Gramian Based Model Reduction

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Bifurcation and Model Reduction Techniques for Large Multi-Disciplinary Systems
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Projection Methods for MOR

Brief Intro to Gramian Based Model Reduction

Balanced Reduction

Balanced Reduction of Oseen Eqns: Extension to Descriptor System

Solving Large Descriptor Lyapunov Equations:
Approximate Balancing - Overcoming Singularities

Neural Modeling: Local Reduction $\Rightarrow$ Many Interactions

Nonlinear MOR: Experiments with EIM

LTI Model Reduction by Projection

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \end{aligned}$$

Approximate $\mathbf{x} \in \mathcal{S}_V = Range(\mathbf{V})$, a k-diml. subspace
i.e. Put $\mathbf{x} = \mathbf{V}\hat{\mathbf{x}}$, and then force

$$\begin{aligned} \mathbf{W}^T[\mathbf{V}\dot{\hat{\mathbf{x}}} &- (\mathbf{A}\mathbf{V}\hat{\mathbf{x}} + \mathbf{B}\mathbf{u})] = 0 \\ \hat{\mathbf{y}} &= \mathbf{C}\mathbf{V}\hat{\mathbf{x}} \end{aligned}$$

If $\mathbf{W}^T\mathbf{V} = \mathbf{I}_k$, then the k dimensional reduced model is

$$\begin{aligned} \dot{\hat{\mathbf{x}}} &= \hat{\mathbf{A}}\hat{\mathbf{x}} + \hat{\mathbf{B}}\mathbf{u} \\ \hat{\mathbf{y}} &= \hat{\mathbf{C}}\hat{\mathbf{x}} \end{aligned}$$

where $\hat{\mathbf{A}} = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\hat{\mathbf{B}} = \mathbf{W}^T\mathbf{B}$, $\hat{\mathbf{C}} = \mathbf{C}\mathbf{V}$.

Moment Matching ↔ Krylov Subspace Projection

Based on Lanczos, Arnoldi, Rational Krylov methods

Padé via Lanczos (PVL)

Freund, Feldmann

Bai

Multipoint Rational Interpolation

Grimme

Gallivan, Grimme, Van Dooren

Recent: Optimal $\mathcal{H}_2$ approximation via interpolation
Gugercin, Antoulas, Beattie

Gramian Based Model Reduction

Proper Orthogonal Decomposition (POD)
Principal Component Analysis (PCA)

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{y} = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t))$$

The Gramian

$$\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau$$

Eigenvectors of $\mathcal{P}$

$$\mathcal{P} = \mathbf{V}\mathbf{S}^2\mathbf{V}^T$$

Orthogonal Basis

$$\mathbf{x}(t) = \mathbf{V}\mathbf{S}\mathbf{w}(t)$$

PCA or POD Reduced Basis

Low Rank Approximation

$$\mathbf{x} \approx \mathbf{V}_k \hat{\mathbf{x}}_k(t)$$

Galerkin condition – Global Basis

$$\dot{\hat{\mathbf{x}}}_k = \mathbf{V}_k^T \mathbf{f}(\mathbf{V}_k \hat{\mathbf{x}}_k(t), \mathbf{u}(t))$$

Global Approximation Error ? ($\mathcal{H}_2$ bound for LTI)

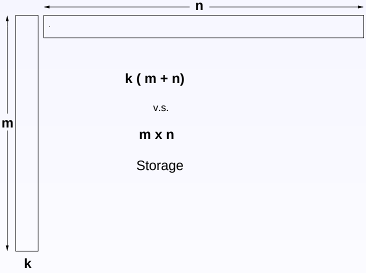
$$\|\mathbf{x} - \mathbf{V}_k \hat{\mathbf{x}}_k\|_2 \approx \sigma_{k+1}$$

Snapshot Approximation to $\mathcal{P}$

$$\mathcal{P} \approx \frac{1}{m} \sum_{j=1}^{m} \mathbf{x}(t_j)\mathbf{x}(t_j)^T = \mathbf{X}\mathbf{X}^T$$

Truncate SVD : $\mathbf{X} = \mathbf{V}\mathbf{S}\mathbf{U}^T \approx \mathbf{V}_k \mathbf{S}_k \mathbf{U}_k^T$

SVD Compression



Advantage of SVD Compression

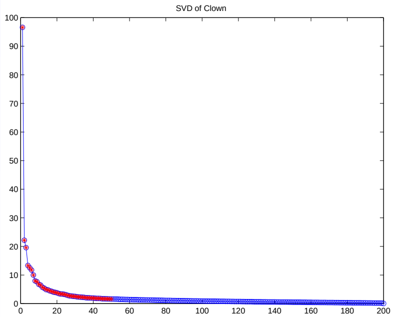
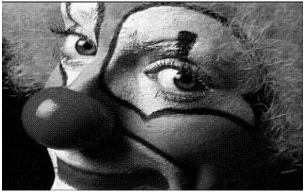


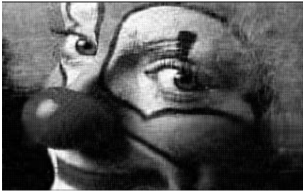
Image Compression - Feature Detection

original

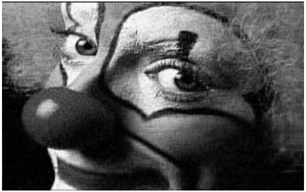


rank = 10

rank = 30



rank = 50



POD in CFD

Extensive Literature

Karhunen-Loéve, L. Sirovich

Burns, King

Kunisch and Volkwein

Gunzburger

Many, many others

Incorporating Observations – Balancing

Lall, Marsden and Glavaski

K. Willcox and J. Peraire

POD for LTI systems

Impulse Response: $\mathcal{H}(t) = \mathbf{C}(t\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}, \quad t \geq 0$

Input to State Map: $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{B}$

Controllability Gramian:

$$\mathcal{P} = \int_o^\infty \mathbf{x}(\tau)\mathbf{x}(\tau)^T d\tau = \int_o^\infty e^{\mathbf{A}\tau}\mathbf{B}\mathbf{B}^T e^{\mathbf{A}^T\tau} d\tau$$

State to Output Map: $\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0)$

Observability Gramian:

$$\mathcal{Q} = \int_o^\infty e^{\mathbf{A}^T\tau}\mathbf{C}^T\mathbf{C}e^{\mathbf{A}\tau} d\tau$$

Balanced Reduction (Moore 81)

Lyapunov Equations for system Gramians

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

With $\mathcal{P} = \mathcal{Q} = \mathbf{S}$: Want Gramians Diagonal and Equal

States Difficult to Reach are also Difficult to Observe

Reduced Model $\mathbf{A}_k = \mathbf{W}_k^T\mathbf{A}\mathbf{V}_k$, $\mathbf{B}_k = \mathbf{W}_k^T\mathbf{B}$, $\mathbf{C}_k = \mathbf{C}_k\mathbf{V}_k$

- $\mathcal{P}\mathbf{V}_k = \mathbf{W}_k\mathbf{S}_k$ $\qquad$ $\mathcal{Q}\mathbf{W}_k = \mathbf{V}_k\mathbf{S}_k$
- Reduced Model Gramians $\mathcal{P}_k = \mathbf{S}_k$ and $\mathcal{Q}_k = \mathbf{S}_k$.

Hankel Norm Error estimate (Glover 84)

Why Balanced Truncation?

- Hankel singular values $= \sqrt{\lambda(\mathcal{P}\mathcal{Q})}$
- Model reduction $\mathcal{H}_\infty$ error (Glover)

  $\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$

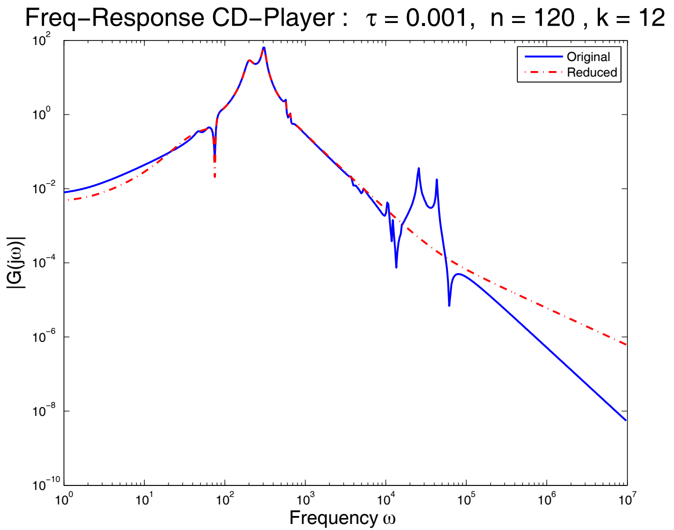
- Extends to MIMO
- Preserves Stability

Key Challenge

- Approximately solve large scale Lyapunov Equations
  in Low Rank Factored Form

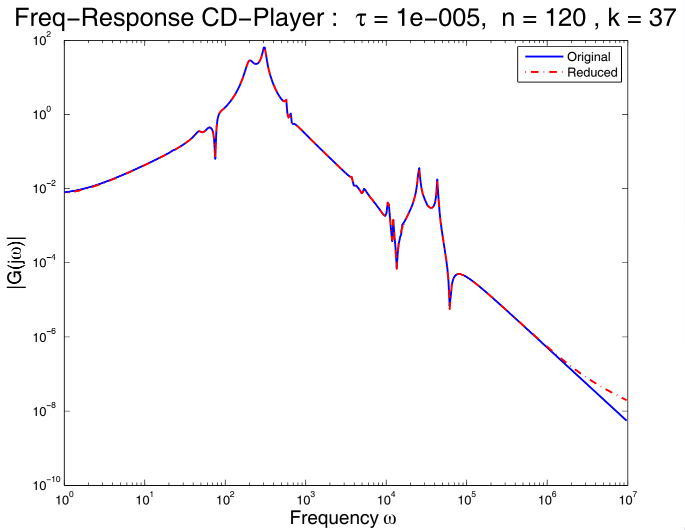
CD Player Frequency Response

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values}) \|u\|_2$$



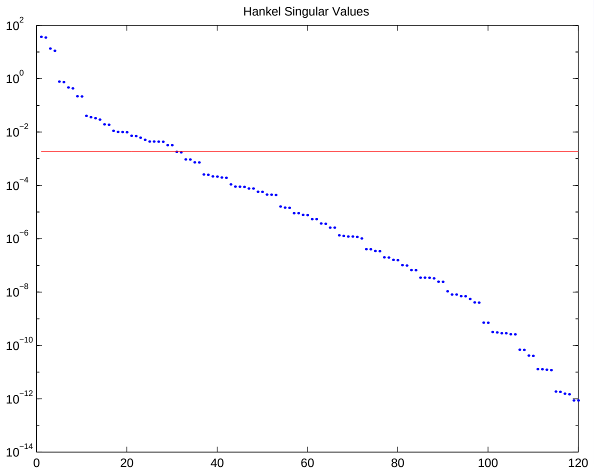
CD Player Frequency Response

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$



CD Player - Hankel Singular Values $\sqrt{\lambda(\mathcal{PQ})}$

$$\|\mathbf{y} - \hat{\mathbf{y}}\|_2 \leq 2 \times (\text{sum neglected singular values})\|u\|_2$$



Approximate Balancing

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \mathbf{A}^T\mathcal{Q} + \mathcal{Q}\mathbf{A} + \mathbf{C}^T\mathbf{C} = 0$$

- Sparse Case: Iteratively Solve in Low Rank Factored Form,

$$\mathcal{P} \approx \mathbf{U}_k\mathbf{U}_k^T, \quad \mathcal{Q} \approx \mathbf{L}_k\mathbf{L}_k^T$$

$$[\mathbf{X}, \mathbf{S}, \mathbf{Y}] = \mathrm{svd}(\mathbf{U}_k^T\mathbf{L}_k)$$

$$\mathbf{W}_k = \mathbf{L}\mathbf{Y}_k\mathbf{S}_k^{-1/2} \text{ and } \mathbf{V}_k = \mathbf{U}\mathbf{X}_k\mathbf{S}_k^{-1/2}.$$

Now: $\mathcal{P}\mathbf{W}_k \approx \mathbf{V}_k\mathbf{S}_k$ and $\mathcal{Q}\mathbf{V}_k \approx \mathbf{W}_k\mathbf{S}_k$

Low Rank Smith = ADI

Convert to Stein Equation:

$$\mathbf{A}\mathcal{P} + \mathcal{P}\mathbf{A}^T + \mathbf{B}\mathbf{B}^T = 0 \quad \Longleftrightarrow \quad \mathcal{P} = \mathbf{A}_\mu \mathcal{P} \mathbf{A}_\mu^T + \mathbf{B}_\mu \mathbf{B}_\mu^T,$$

where

$$\mathbf{A}_\mu = (\mathbf{A} - \mu\mathbf{I})(\mathbf{A} + \mu\mathbf{I})^{-1}, \quad \mathbf{B}_\mu = \sqrt{2|\mu|}(\mathbf{A} + \mu\mathbf{I})^{-1}\mathbf{B}.$$

Solution:

$$\mathcal{P} = \sum_{j=0}^{\infty} \mathbf{A}_\mu^j \mathbf{B}_\mu \mathbf{B}_\mu^T (\mathbf{A}_\mu^j)^T = \mathbf{L}\mathbf{L}^T,$$

where $\mathbf{L} = [\mathbf{B}_\mu,\ \mathbf{A}_\mu\mathbf{B}_\mu,\ \mathbf{A}_\mu^2\mathbf{B}_\mu,\ \ldots]$ Factored Form

Multi-Shift (Modified) Low Rank Smith

LR - Smith: Update Factored Form $\mathcal{P}_m = L_m L_m^T$:
(*Penzl*, *Li*, *White*)

$$\begin{aligned} \mathbf{L}_{m+1} &= [\mathbf{A}_\mu \mathbf{L}_m, \mathbf{B}_\mu] \\ &= [\mathbf{A}_\mu^{m+1} \mathbf{B}_\mu, \mathbf{L}_m] \end{aligned}$$

Multi-Shift LR - Smith: (Gugercin, Antoulas, and S.)

- Update and Truncate SVD
- Re-Order and Aggregate Shift Applications
- Much Faster and Far Less Storage

$$\begin{aligned} \mathbf{B} &\leftarrow \mathbf{A}_\mu \mathbf{B}; \\ [\mathbf{V}, \mathbf{S}, \mathbf{Q}] &= \mathrm{svd}([\mathbf{A}_\mu \mathbf{B}, \mathbf{L}_\mathrm{m}]); \\ \mathbf{L}_{m+1} &\leftarrow \mathbf{V}_k \mathbf{S}_k; \quad (\sigma_{k+1} < tol \cdot \sigma_1) \end{aligned}$$

Balanced Truncation MOR of Oseen Eqn.

Semi-Discrete Oseen Equations: A Descriptor System

$$\begin{aligned}
\mathbf{E}_{11}\frac{d}{dt}\mathbf{v}(t) &= \mathbf{A}_{11}\mathbf{v}(t) + \mathbf{A}_{12}\mathbf{p}(t) + \mathbf{B}_1\mathbf{g}(t), \\
\mathbf{0} &= \mathbf{A}_{12}^T\mathbf{v}(t), \\
\mathbf{v}(0) &= \mathbf{v}_0, \\
\mathbf{y}(t) &= \mathbf{C}_1\mathbf{v}(t) + \mathbf{C}_2\mathbf{p}(t) + \mathbf{D}\mathbf{g}(t).
\end{aligned}$$

$$\mathcal{E}\frac{d}{dt}\mathbf{v}(t) = \mathcal{A}\mathbf{v}(t) + \mathcal{B}\mathbf{u}(t), \qquad \mathbf{y}(t) = \mathcal{C}\mathbf{v}(t) + \mathcal{D}\mathbf{u}(t)$$

Note $\mathcal{E}$ is <u>singular</u>   index 2

See Stykel (LAA 06) – general theory and approach

Eliminate Pressure

- Incompressibility eqn.: $\mathbf{0} = \mathbf{A}_{12}^T \mathbf{v}(t) \quad \Rightarrow \quad \mathbf{A}_{12}^T \frac{d}{dt}\mathbf{v}(t) = \mathbf{0}$.
- Use to eliminate pressure from velocity ODE
- Obtain projected velocity ODE

$$\mathbf{E}_{11} \frac{d}{dt}\mathbf{v}(t) = \mathbf{\Pi}\mathbf{A}_{11}\mathbf{v}(t) + \mathbf{\Pi}\mathbf{B}_1\mathbf{g}(t),$$

- $\mathbf{\Pi}$ is an oblique projection: $\mathbf{\Pi}^2 = \mathbf{\Pi}$, $\mathbf{\Pi}\mathbf{E}_{11} = \mathbf{E}_{11}\mathbf{\Pi}^T$, $\mathrm{null}(\mathbf{\Pi}) = \mathrm{range}(\mathbf{A}_{12})$ and $\mathrm{range}(\mathbf{\Pi}) = \mathrm{null}(\mathbf{A}_{12}^T\mathbf{E}_{11}^{-1})$. In particular,

$$\mathbf{A}_{12}^T \mathbf{z} = \mathbf{0} \quad \text{if and only if} \quad \mathbf{\Pi}^T \mathbf{z} = \mathbf{z}.$$

Notation

Put

$$\widetilde{\mathbf{E}} = \mathbf{\Pi}\mathbf{E}_{11}\mathbf{\Pi}^T, \quad \widetilde{\mathbf{A}} = \mathbf{\Pi}\mathbf{A}_{11}\mathbf{\Pi}^T, \quad \widetilde{\mathbf{B}} = \mathbf{\Pi}\mathbf{B}_1, \quad \widetilde{\mathbf{C}} = \mathbf{C}\mathbf{\Pi}^T.$$

With this notation,

$$\begin{aligned} \widetilde{\mathbf{A}}\,\mathbf{P}\,\widetilde{\mathbf{E}} + \widetilde{\mathbf{E}}\,\mathbf{P}\,\widetilde{\mathbf{A}}^T + \widetilde{\mathbf{B}}\widetilde{\mathbf{B}}^T &= \mathbf{0}, \\ \widetilde{\mathbf{A}}^T\,\mathbf{Q}\,\widetilde{\mathbf{E}} + \widetilde{\mathbf{E}}\,\mathbf{Q}\,\widetilde{\mathbf{A}} + \widetilde{\mathbf{C}}^T\widetilde{\mathbf{C}} &= \mathbf{0}. \end{aligned}$$

where

$$\begin{aligned} \mathbf{\Pi} &= \mathbf{I} - \mathbf{A}_{12}(\mathbf{A}_{12}^T\mathbf{E}_{11}^{-1}\mathbf{A}_{12})^{-1}\mathbf{A}_{12}^T\mathbf{E}_{11}^{-1} \\ &= \mathbf{\Theta}_l\mathbf{\Theta}_r^T \end{aligned}$$

$\mathbf{\Pi}^T$ Projector onto Null($\mathbf{A}_{12}^T$)

Projected Stein Equation

$$\mathbf{P} = \widetilde{\mathbf{A}}_\mu \mathbf{P} \widetilde{\mathbf{A}}_\mu^* - 2\,\mathrm{Re}(\,\mu) \widetilde{\mathbf{B}}_\mu \widetilde{\mathbf{B}}_\mu^*.$$

where

$$\widetilde{\mathbf{A}}_\mu \equiv \left(\widetilde{\mathbf{E}} + \mu\,\widetilde{\mathbf{A}}\right)^{I} \left(\widetilde{\mathbf{E}} - \bar{\mu}\,\widetilde{\mathbf{A}}\right), \quad \text{and} \quad \widetilde{\mathbf{B}}_\mu \equiv \left(\widetilde{\mathbf{E}} + \mu\,\widetilde{\mathbf{A}}\right)^{I} \widetilde{\mathbf{B}}$$

Solution:

$$\mathbf{P} = -2\,\mathrm{Re}(\mu) \sum_{j=0}^{\infty} \widetilde{\mathbf{A}}_\mu^j \widetilde{\mathbf{B}}_\mu \widetilde{\mathbf{B}}_\mu^* \left(\widetilde{\mathbf{A}}_\mu^*\right)^j .$$

Convergent for stable pencil with $\mathrm{Real}(\mu) < 0$

Key Implementation Lemma

If $\mathbf{M} = \mathbf{\Pi}^T \mathbf{M}$, then the computation

$$\mathbf{Z} = \left(\widetilde{\mathbf{E}} + \mu \widetilde{\mathbf{A}}\right)^I \left(\widetilde{\mathbf{E}} - \bar{\mu} \widetilde{\mathbf{A}}\right) \mathbf{M}$$

may be accomplished with the following steps.

1. Put $\mathbf{F} = (\mathbf{E}_{11} - \bar{\mu} \mathbf{A}_{11}) \mathbf{M}$.
2. Solve

$$\begin{pmatrix} \mathbf{E}_{11} + \mu \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{0} \end{pmatrix}.$$

Note that $\mathbf{Z}$ satisfies $\mathbf{Z} = \mathbf{\Pi}^T \mathbf{Z}$
Similar result holds for computing $\widetilde{\mathbf{B}}_\mu$

Algorithm:Single Shift ADI

1. Solve $\begin{pmatrix} \mathbf{E}_{11} + \mu\,\mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix}$;
2. $\mathbf{U} = \mathbf{Z}$;
3. **while** ('not converged')
   3.1 $\mathbf{Z} \leftarrow (\mathbf{E}_{11} - \bar{\mu}\,\mathbf{A}_{11})\,\mathbf{Z}$;
   3.2 Solve (in place) $\begin{pmatrix} \mathbf{E}_{11} + \mu\,\mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{Z} \\ \mathbf{0} \end{pmatrix}$;
   3.3 $\mathbf{U} \leftarrow [\mathbf{U}, \mathbf{Z}]$;

   end
4. $\mathbf{U} \leftarrow \sqrt{2|\,\mathrm{Re}(\mu)|}\,\mathbf{U}$.

Algorithm:Multi-Shift ADI

1. $\mathbf{U} = [\ \ ]$;
2. **while** ('not converged')

    for i = 1:m,

    2.1 Solve $\begin{pmatrix} \mathbf{E}_{11} + \mu_i \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ \mathbf{0} \end{pmatrix}$;

    2.2 $\mathbf{U}_0 = \mathbf{Z}$;

    2.3 **for** j = 1:k-1,

    2.3.1 $\mathbf{Z} \leftarrow (\mathbf{E}_{11} - \bar{\mu}_i \mathbf{A}_{11})\, \mathbf{Z}$;

    2.3.2 Solve (in place) $\begin{pmatrix} \mathbf{E}_{11} + \mu_i \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{12}^T & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{Z} \\ \mathbf{\Lambda} \end{pmatrix} = \begin{pmatrix} \mathbf{Z} \\ \mathbf{0} \end{pmatrix}$;

    2.3.3 $\mathbf{U}_0 \leftarrow [\mathbf{U}_0, \mathbf{Z}]$;

    end

    2.4 $\mathbf{U} \leftarrow \left[\mathbf{U}, \sqrt{2|\ \mathrm{Re}(\mu_i)|}\, \mathbf{U}_0\right]$;

    % *Update and truncate* SVD($\mathbf{U}$);

    2.5 $\mathbf{B} \leftarrow (\mathbf{E}_{11} - \bar{\mu}_i \mathbf{A}_{11})\, \mathbf{Z}$.

    end

end

Model Problem: Oseen Equations

$$
\begin{aligned}
\frac{\partial}{\partial t} v(x,t) + (a(x)\cdot\nabla) v(x,t) &- \nu \Delta v(x,t) + \nabla p(x,t) \\
&= \chi_{\Omega_g}(x) g_\Omega(x,t) \quad \text{in } \Omega \times (0,T), \\
\nabla \cdot v(x,t) &= 0 \quad \text{in } \Omega \times (0,T), \\
(-p(x,t)I + \nu \nabla v(x,t)) n(x) &= 0 \quad \text{on } \Gamma_n \times (0,T), \\
v(x,t) &= 0 \quad \text{on } \Gamma_d \times (0,T), \\
v(x,t) &= g_\Gamma(x,t) \quad \text{on } \Gamma_g \times (0,T), \\
v(x,0) &= v_0(x) \quad \text{in } \Omega,
\end{aligned}
$$

Channel Geometry and Grid

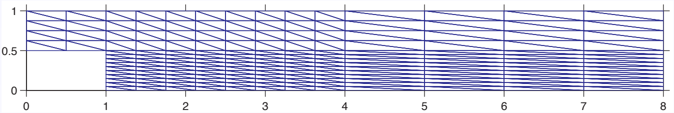


Figure: The channel geometry and coarse grid

EXAMPLE 1.

$$\mathbf{y}(t) = \int_{\Omega_{\mathrm{obs}}} -\partial_{x_2} v_1(x,t) + \partial_{x_1} v_2(x,t) dx$$

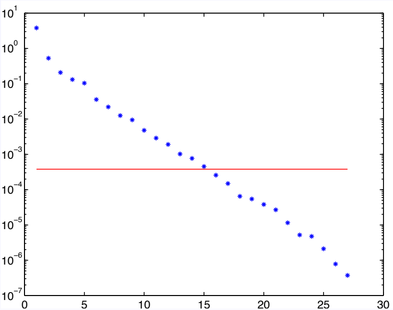
over the subdomain $\Omega_{\mathrm{obs}} = (1,3) \times (0,1/2)$.

Model Reduction Results

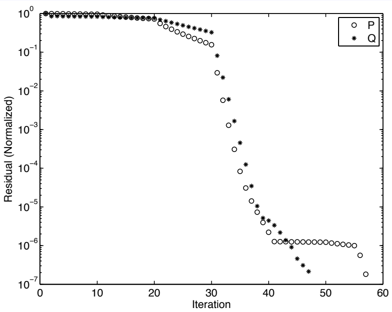
n_v	n_p	k
1352	205	13
5520	761	14
12504	1669	15
22304	2929	15

Table: Number n_v of semidiscrete velocities $\mathbf{v}(t)$, number n_p of semidiscrete pressures $\mathbf{p}(t)$, and and size k of the reduced order velocities $\widehat{\mathbf{v}}(t)$ for various uniform refinements of the coarse grid.

Hankel S-vals and Convergence ADI



The largest Hankel singular values



Convergence of the multishift ADI Algorithm

Figure: The left plot shows the largest Hankel singular values and the threshold $\tau\sigma_1$. The right plot shows the normalized residuals $\|\widetilde{\mathbf{B}}_k\|_2$ generated by the multishift ADI Algorithm . for the approximate solution of the controllability Lyapunov equation ($\circ$) and of the observability Lyapunov equation ($*$).

Time (left) and Frequency (right) Responses

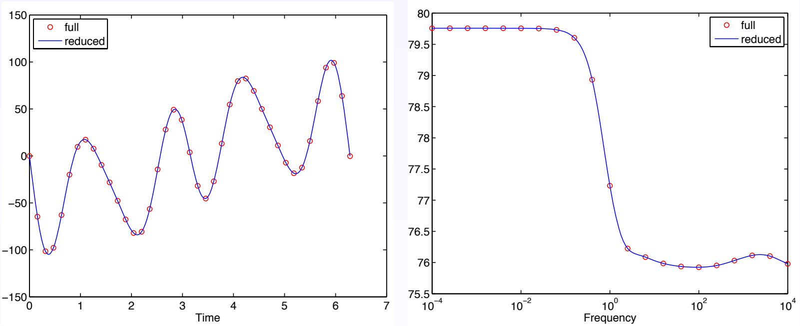


Figure: Time response (left) and frequency response (right) for the full order model (circles) and for the reduced order model (solid line).

Velocity Profile

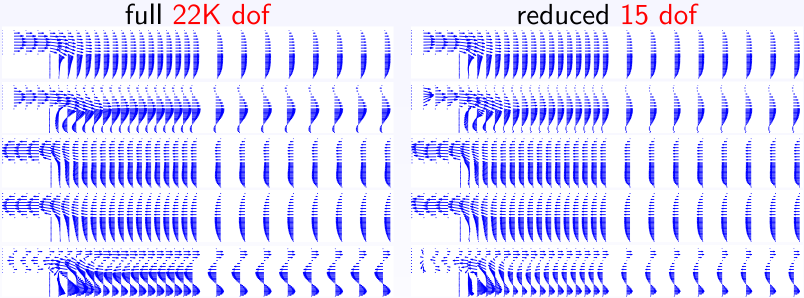


Figure: Velocities generated with the full order model (left column) and with the reduced order model (right column) at $t = 1.0996, 2.9845, 3.7699, 4.86965, 6.2832$ (top to bottom).

Balanced Truncation on a Compartmental Neuron Model

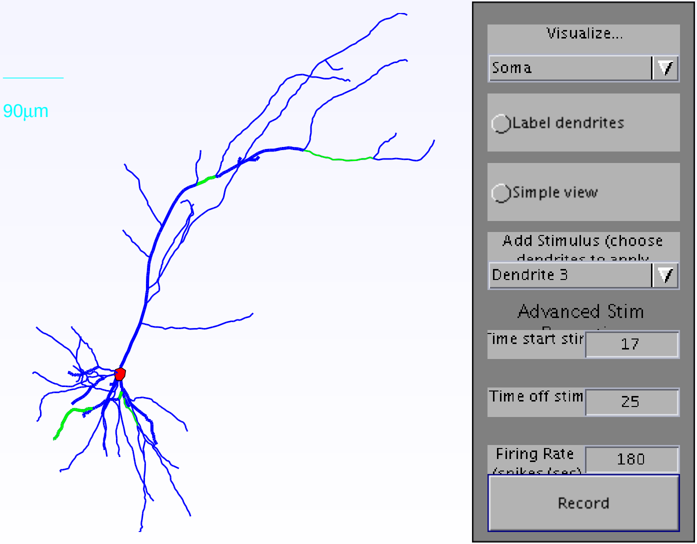
Steve Cox

Tony Kellems

Undergrads Nan Xiao and Derrick Roos

Quasi-active integrate and fire model.
Complex model dimension 6000
Faithfully approximated with 10-20 variable ROM

Neuron Cell


90µm
Visualize...
Soma
Label dendrites
Simple view
Add Stimulus (choose
Dendrite 3
Advanced Stim
Time start stir
17
Time off stim
25
Firing Rate
180
Record

Full Non-Linear Model

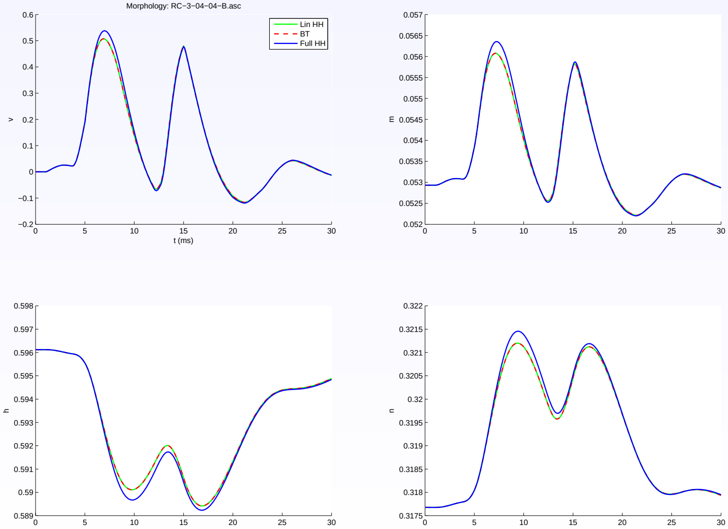
$I_{j,syn}$ is the synaptic input into branch j

$$\frac{a_j}{2R_i}\partial_{xx} v_j = C_m \partial_t v_j \quad + \quad G_{Na} m_j^3 h_j (v_j - E_{Na})$$
$$+ \quad G_K n_j^4 (v_j - E_K) + G_l (v_j - E_l) + I_{j,syn}(x,t)$$

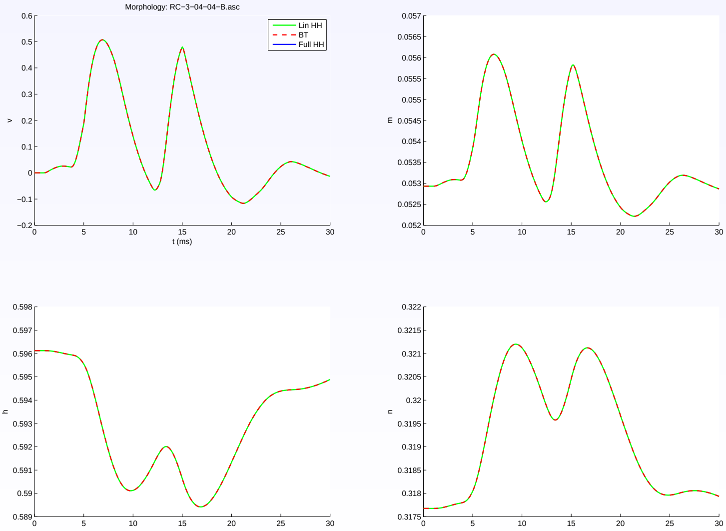
Kinetics of the potassium (n) and sodium (h, m) channels

$$\begin{aligned} \partial_t m_j &= \alpha_m(v_j)(1 - m_j) - \beta_m(v_j) m_j \\ \partial_t h_j &= \alpha_h(v_j)(1 - h_j) - \beta_h(v_j) h_j \\ \partial_t n_j &= \alpha_n(v_j)(1 - n_j) - \beta_n(v_j) n_j. \end{aligned}$$

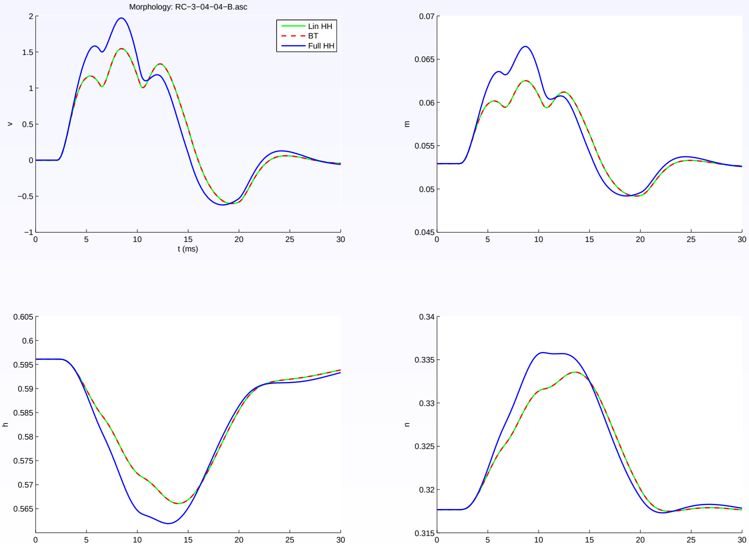
Cell Response - Lin and Non-Lin



Cell Response - Linear



Cell Response - Near Threshold



Optimal $\mathcal{H}_2$ Methods: IRKA

Kellums and Nong

Using Variant of IRKA

Gugercin, Antoulas, Beattie (2008)

Reduced 1.6M Neuron Model to a ROM of order 20.

Solve times to construct ROM are under 2 mins
(High End Workstation)

Parameter Study Experiment 158 hrs (full) $\rightarrow$ 3.4 hrs

Neural ROM Results

- Interesting example of many-input , single-output system
- Simulation time single cell 14 sec Full vs .01 sec Reduced
- Ultimate goal is to simulate a few-Million neuron system over a minute of brain-time: Feasibility Demonstrated
- Currently limited to a 10K neuron system over a few brain-seconds without new technology
- Parallel computing required

Model Reduction of Nonlinear Terms

Saifon Chaturantabut

Implemenation of EIM method of Patera et.al. (2004)

Model Problem: Unsteady 1D Burgers' Equation (K. Willcox)

$$\frac{\partial}{\partial t}y(x,t) - \nu\frac{\partial^2}{\partial x^2}y(x,t) + \frac{\partial}{\partial x}\left(\frac{y(x,t)^2}{2}\right) = 0 \qquad x \in [0,1], t \geq 0$$

$$\begin{aligned} y(0,t) = y(1,t) &= 0 \qquad t \geq 0 \\ y(x,0) &= y_0(x) \qquad x \in [0,1], \end{aligned}$$

- Commonly used model problem to test algorithms for Navier-Stokes type equations
  - Simple nonlinearity that is sufficient to test algorithm

Direct POD

If we apply the POD basis directly to construct a discretized system, the original system of order N:

$$\mathbf{M}_h \frac{d}{dt}\mathbf{y}(t) + \nu \mathbf{K}_h \mathbf{y}(t) - \mathcal{N}_h(\mathbf{y}(t)) = 0$$

become a system of order $k \ll N$:

$$\tilde{\mathbf{M}} \frac{d}{dt}\tilde{\mathbf{y}}(t) + \nu \tilde{\mathbf{K}} \tilde{\mathbf{y}}(t) - \tilde{\mathcal{N}}(\tilde{\mathbf{y}}(t)) = 0,$$

where the **nonlinear term :**

$$\boxed{\tilde{\mathcal{N}}(\tilde{\mathbf{y}}(t)) = \underbrace{\mathbf{U}^T}_{k \times N} \underbrace{\mathcal{N}_h(\mathbf{U}\tilde{\mathbf{y}}(t))}_{N \times 1}}$$

$\Rightarrow$ Computational Complexity still depends on N!!

Nonlinear Approximation via EIM

WANT:

$$\boxed{\tilde{\mathcal{N}}(\tilde{\mathbf{y}}(t)) \leftarrow \underbrace{\mathbf{C}}_{k \times n_m} \underbrace{\hat{\mathcal{N}}(\tilde{\mathbf{y}}(t))}_{n_m \times 1}} \dashrightarrow \boxed{k, n_m \ll N} \dashrightarrow \text{Indep. of } N$$

Nonlinear Approximation via EIM Contd.

HOW:

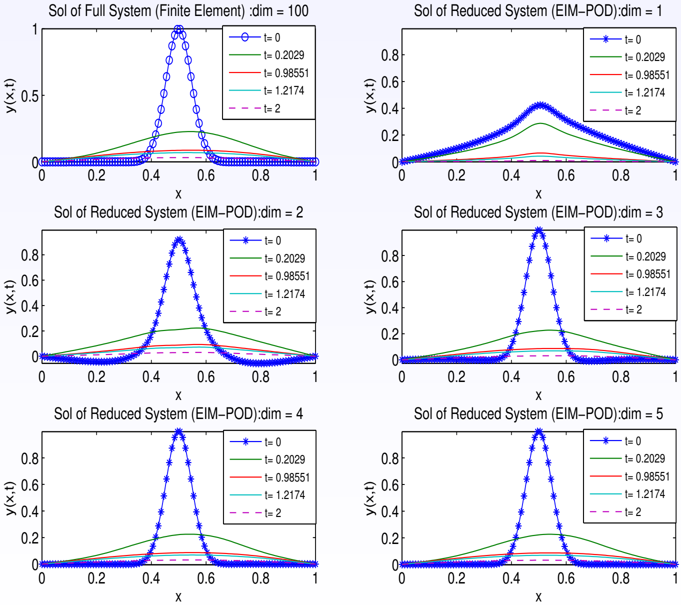
$$\mathbf{C}\hat{\mathcal{N}}(\tilde{\mathbf{y}}(t)) = \underbrace{\left(\mathbf{U}^T \int_\Omega [\Psi(x)']^T \mathcal{Q}(x)\mathrm{d}x\right)}_{\mathbf{C}} \underbrace{\left(\mathcal{Q}_{\mathbf{z}}^{-1}\mathbf{s}_{\mathbf{z}}(t)\right)}_{\hat{\mathcal{N}}(\tilde{\mathbf{y}}(t))}$$

where $s(y) = y^2$ and

- $\mathbf{s}_{\mathbf{z}}(t) = [s(\Phi(z_1)\tilde{\mathbf{y}}(t)), s(\Phi(z_2)\tilde{\mathbf{y}}(t)), \ldots s(\Phi(z_{n_m})\tilde{\mathbf{y}}(t))]^T$
- $\{z_j\}$ Empirical Interpolation Points

and where

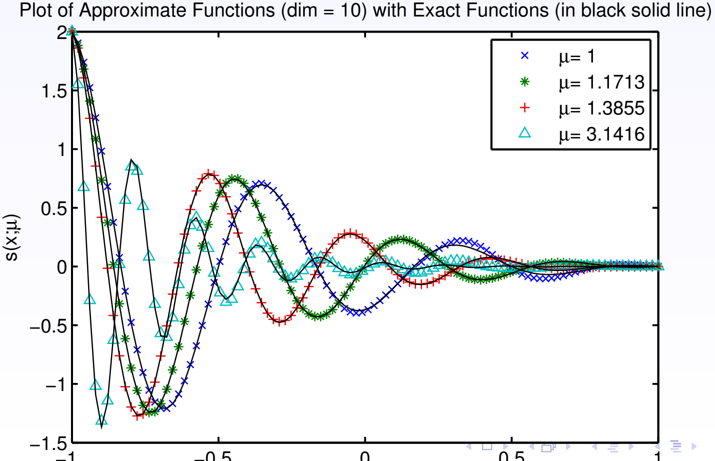
- $\Psi(x) = [\psi_1(x), \psi_2(x), \ldots \psi_N(x)]$ - FEM basis
- $\Phi(x) = [\phi_1(x), \phi_2(x), \ldots \phi_k(x)]$ - POD basis
  - From Snapshots $\Psi(x)\mathbf{y}(t_\ell)$
- $\mathcal{Q}(x) = [q_1(x), q_2(x), \ldots q_{n_m}(x)]$ - POD basis
  - From Snapshots $s(\Psi(x)\mathbf{y}(t_\ell))$


Sol of Full System (Finite Element) :dim = 100
Sol of Reduced System (EIM−POD):dim = 1
Sol of Reduced System (EIM−POD):dim = 2
Sol of Reduced System (EIM−POD):dim = 3
Sol of Reduced System (EIM−POD):dim = 4
Sol of Reduced System (EIM−POD):dim = 5
y(x,t)
x
t= 0
t= 0.2029
t= 0.98551
t= 1.2174
t= 2

EIM: Numerical Example

$$s(x;\mu) = (1-x)cos(3\pi\mu(x+1))e^{-(1+x)\mu},$$

where $x \in [-1, 1]$ and $\mu \in [1, \pi]$.



Summary

Balanced Truncation MOR for a Class of Descriptor Systems
CAAM TR07-02, M. Heinkenschloss, DCS, & K. Sun
(to appear SISC)

Gramian Based Model Reduction: Balanced Reduction

Balanced Reduction of Oseen Eqns: Extension to Descriptor System

Multi-Shift ADI Without Explicit Projectors:
Only need Saddle Point Solver Sparse Direct or Iterative

Neural Modeling - Single Cell ROM $\Rightarrow$ Many Interactions
Example of important class of problems
(including Monte Carlo of Stochastic Systems)

Nonlinear MOR via EIM
Promising approach - feasibility study (Saifon's MS-thesis)