Construction of low–order dynamical models for problems involving non-selfadjoint operators

applied to the salt lake problem

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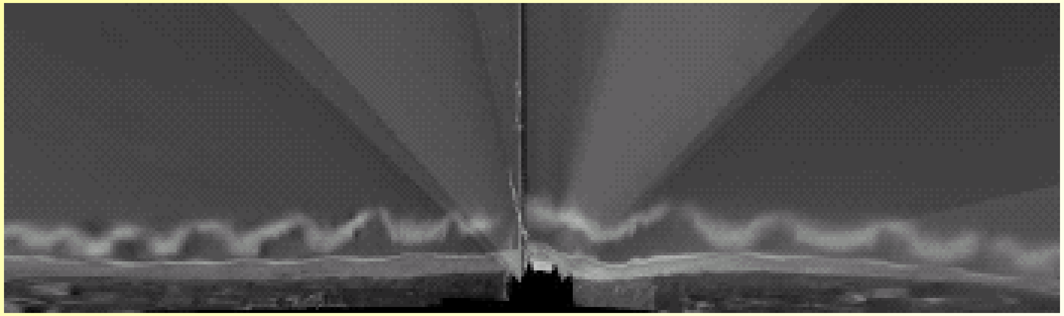
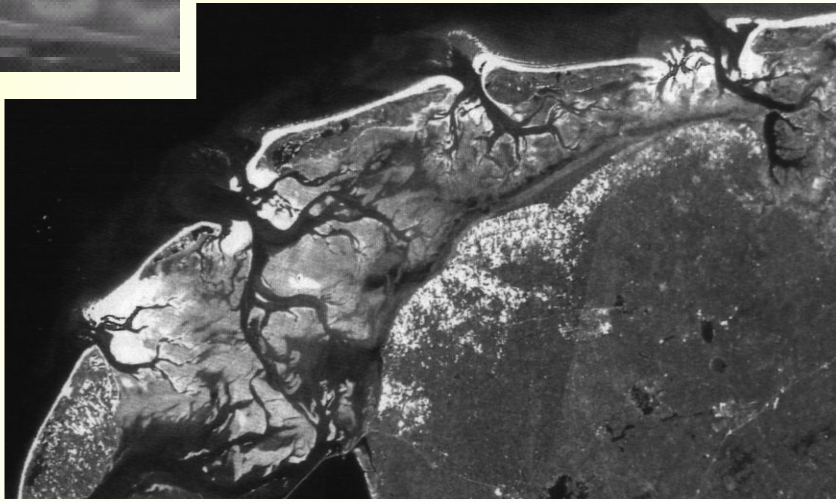
Introduction

• Observations in many natural systems suggest that the dynamics is only governed by a few (interacting) patterns.



• Patterns often resulting from strongly nonlinear interactions (i.e., not close to the onset of linear instability)



Research Question

Can we construct a dynamical model to reproduce, understand and predict the observed dynamical behaviour in an efficient way?

Approach

- Construction of a low-dimensional dynamical model
- Based on a few physically relevant patterns
  physically interpretable patterns
- Can be analysed with well-known mathematical techniques

➡ Choice of patterns is essential!!

Construction of low-dimensional model (1)

Define: state vector $\mathbf{\Phi} = (\ldots)$, i.e. velocity field, saturation, pressure,…

parameter vector $\boldsymbol{\lambda} = (\ldots)$, i.e. evaporation rate, geometry

Dynamics of $\mathbf{\Phi}$:
- coupled system of nonlinear ordinary and partial differential equations
- usually <u>NOT SELF-ADJOINT</u>

$$\mathcal{M} \frac{\partial \mathbf{\Phi}}{\partial t} + \mathcal{L}(\boldsymbol{\lambda})\, \mathbf{\Phi} + \mathcal{N}(\boldsymbol{\lambda}, \mathbf{\Phi}) = \mathbf{F}$$

Where
- $\mathcal{M}$: mass matrix, a linear operator. In many problems $\mathcal{M}$ is singular
- $\mathcal{L}$: linear operator
- $\mathcal{N}$: nonlinear operator
- $\mathbf{F}$: forcing vector

Construction of low-dimensional model (2)

Step 1: identify a steady state solution Φ_{eq} for a certain λ.

$$\mathcal{L}(\lambda)\,\Phi_{eq} + \mathcal{N}(\lambda,\Phi_{eq}) = \mathbf{F}$$

Step 2: investigate the linear stability of Φ_{eq}.

➡ Write $\Phi = \Phi_{eq} + \phi$ and linearize the eqn's:

$$\mathcal{M}\,\frac{\partial\phi}{\partial t} + \mathcal{J}(\lambda)\,\phi = \mathbf{0}$$

with the total jacobian $\mathcal{J} = \mathcal{L}(\lambda) + \mathcal{N}(\lambda,\phi,\Phi_{eq})$
with $\mathcal{N}$ linearized around Φ_{eq}

This generalized eigenvalue-problem (usually solved numerically) gives:

- Eigenvectors r_k
- Adjoint eigenvectors l_k

Construction of low-dimensional model (3)

Step 3: model reduction by Galerkin projection on eigenfunctions.

- Expand ϕ in a FINITE number of eigenfunctions:

$$\phi = \sum_{j=1}^{N} r_j \, a_j(t)$$

- Insert $\Phi = \Phi_{eq} + \phi$ in the equations.
- Project on the adjoint eigenfunctions ➡ evolution equations for the amplitudes $a_j(t)$:

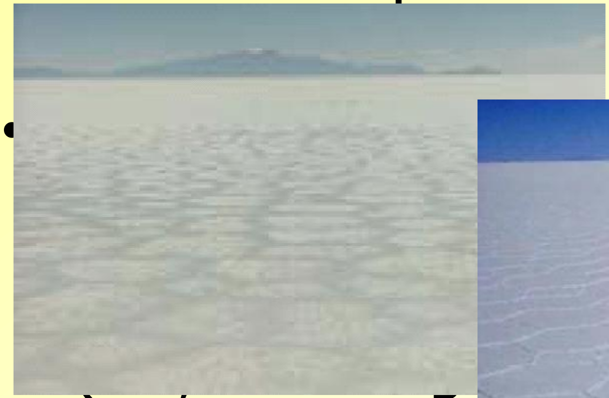
$$a_{j,t} - \sum_{k=1}^{N} \beta_{jk} \, a_k + \sum_{k=1}^{N} \sum_{l=1}^{N} c_{jkl} \, a_k \, a_l = 0, \quad \text{for } j = 1 \ldots N$$

Example of nonlinearity

➡ system of nonlinear PDE’s reduced to a system of coupled ODE’s.

Critical points and choices

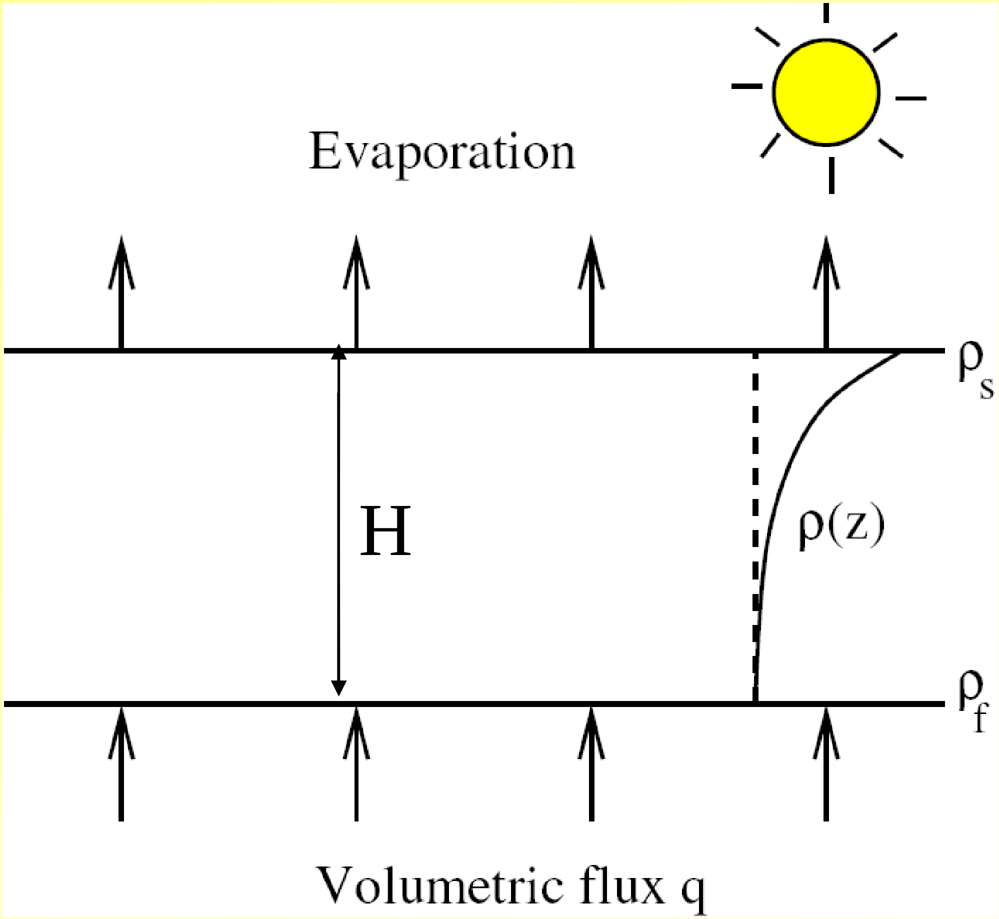
- How ‘good’ is the low-dimensional model?
- Which eigenfunctions should be used to construct the low-dimensional model?
- How many eigenfunctions should be used in the expansion?
- How to keep the low-dimensional



- How persistent forcing by noise.

salt lake problem

Salt lake problem



- Saturation $S = \dfrac{\rho - \rho_f}{\rho_s - \rho_f}$
- Velocity $\mathbf{U} = \dfrac{\mathbf{q}}{u_c}$
- Rayleigh number R:

$$R = \frac{u_c}{\text{Evaporation rate}}$$

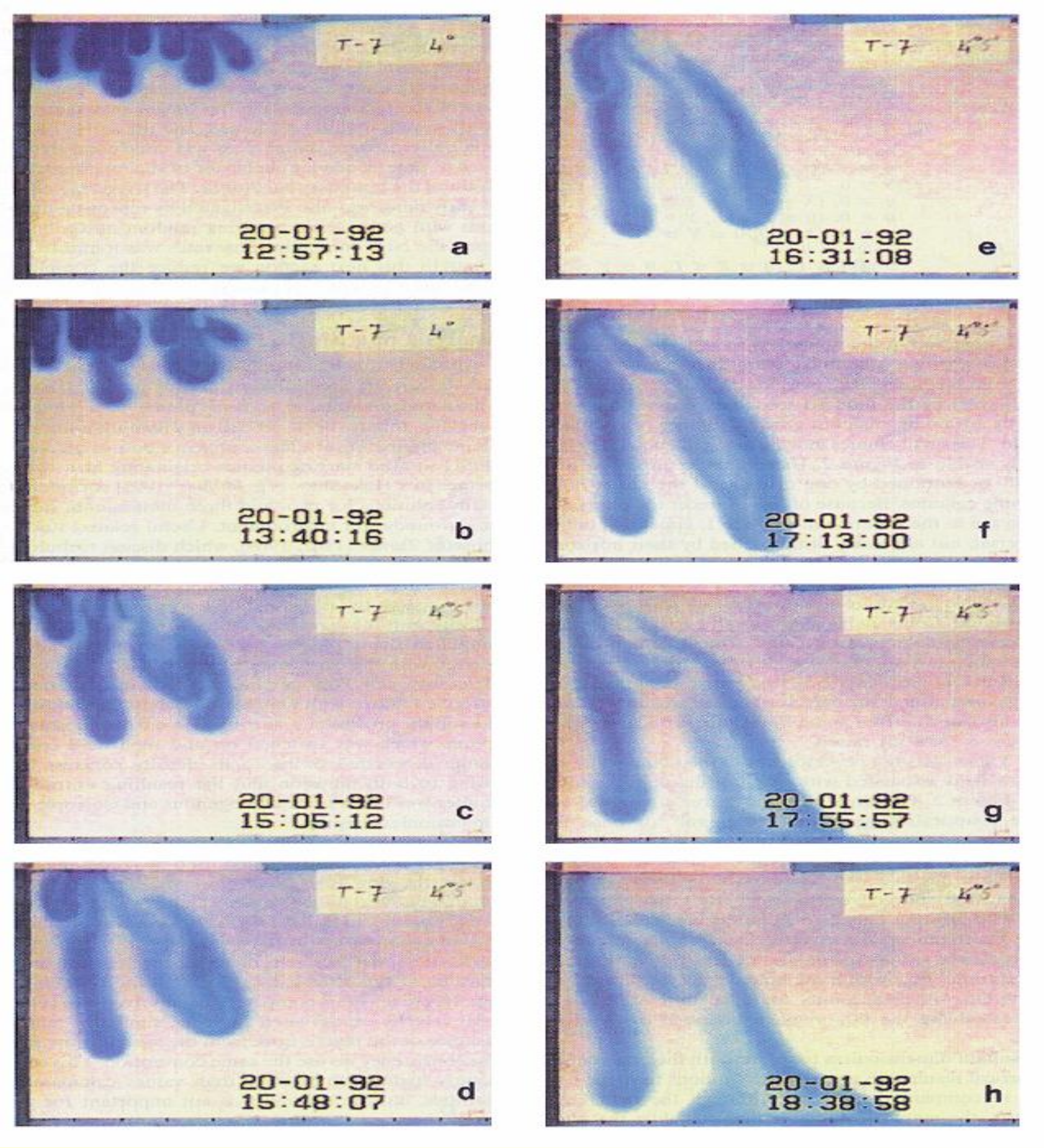
- Peclet number Pe:

$$Pe = \frac{H^{*}\text{Evaporation rate}}{\text{Dispersion coeff}}$$

Lab Experiments (Wooding, 1997) (1)

Initially many fingers

Evolve in one-two fingers



When fingers hit the bottom: complex behaviour

Salt lake problem: model equations

Governing Equations (after scaling):

- $\nabla \cdot \mathbf{U} = 0$ (mass conservation)
- $\mathbf{U} = -(\nabla p - S\,\mathbf{e}_z)$ (Darcy's law)
- $S_t + R\,\nabla \cdot (\mathbf{U}\,S) = Pe^{-1} \Delta S$ (salt mass balance)

Boundary conditions:

- $\mathbf{U} \cdot \mathbf{e}_z = -1/R$ at $z=0,1$
- $S = 1$ at $z=0$
- $S = 0$ at $z=1$
- No-flow b.c. in the vertical plane

Salt lake problem: construction of r.m.(1)

Step 1: Basic state is given by $\mathbf{\Phi}_{eq} = (S,U,p)_{eq} = \mathbf{\Phi}_{eq}(z,R)$

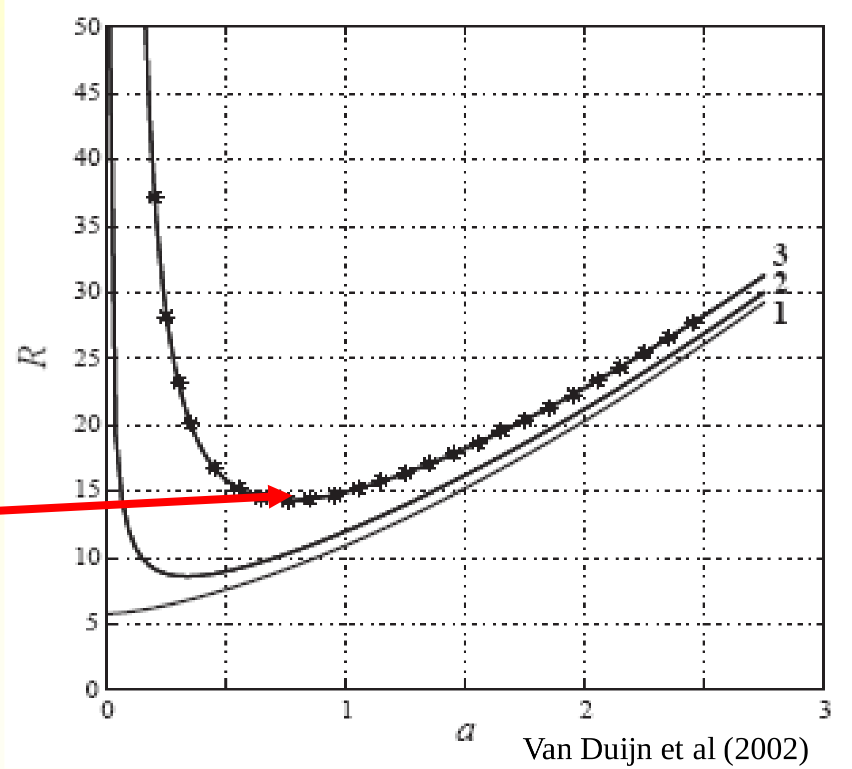
- Uniform upflow
- S exponentially decaying:
- Control parameters R, Pe

$$S_0(z) = \frac{e^{Pe(1-z)} - 1}{e^{Pe} - 1}$$

Step 2: Linear Stability of $\mathbf{\Phi}_{eq}$:

- Write $\Phi = \Phi_{eq} + \varphi$
- Linearize the equations and solve eigenvalue problem

(a_{crit}, R_{crit})

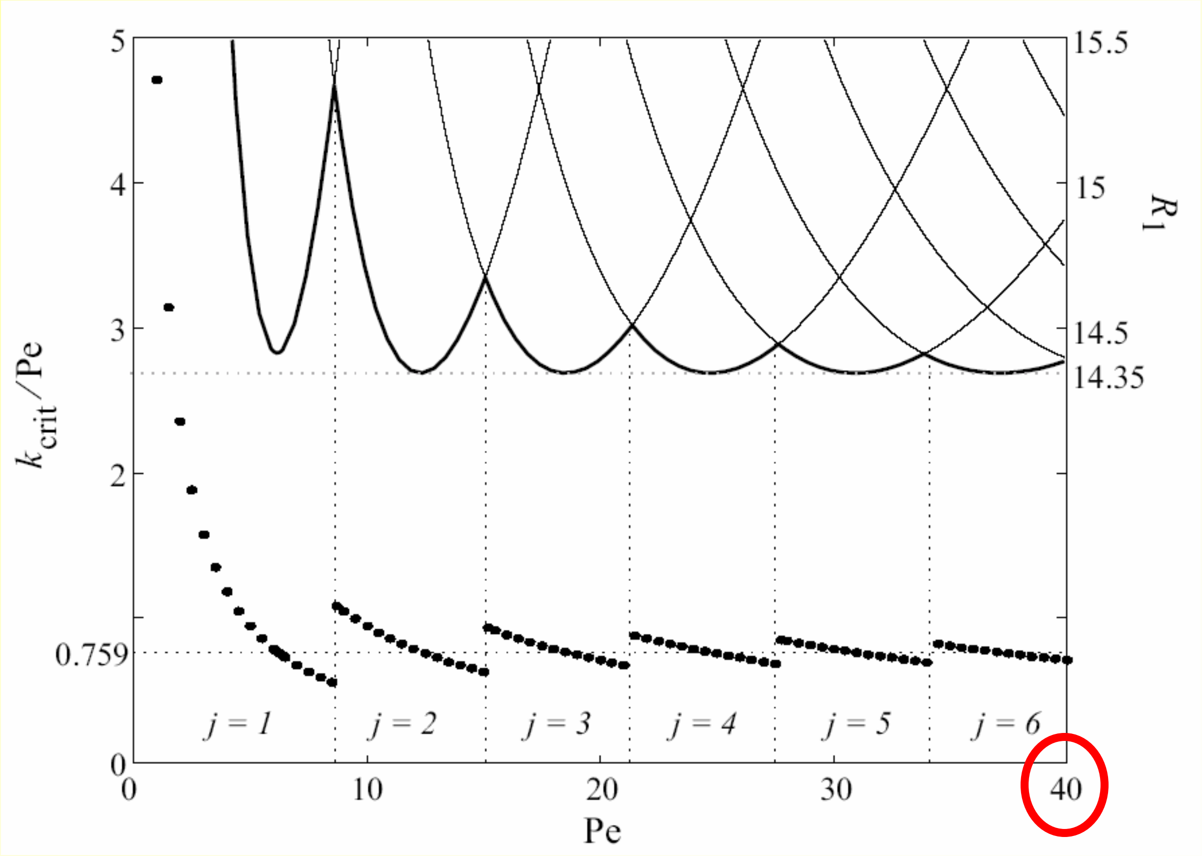


Van Duijn et al (2002)

Salt lake problem: construction of r.m.(2)

Step 3: model reduction by Galerkin projection on eigenfunctions.

- Eigenfunctions calculated at $R=R_{crit}$, patterns kept fixed
- R_{crit} and most unstable pattern depend on Peclet number



Model results

- Bifurcation Structure (Steady States only)

  ➡ Solve the steady state amplitude equations, varying R:

$$\cancel{A_{j,t}} - \sum_{k=1}^{N} \beta_{jk} A_k + \sum_{k=1}^{N} \sum_{l=1}^{N} c_{jkl} A_k A_l = 0, \quad \text{for } j = 1 \ldots N$$

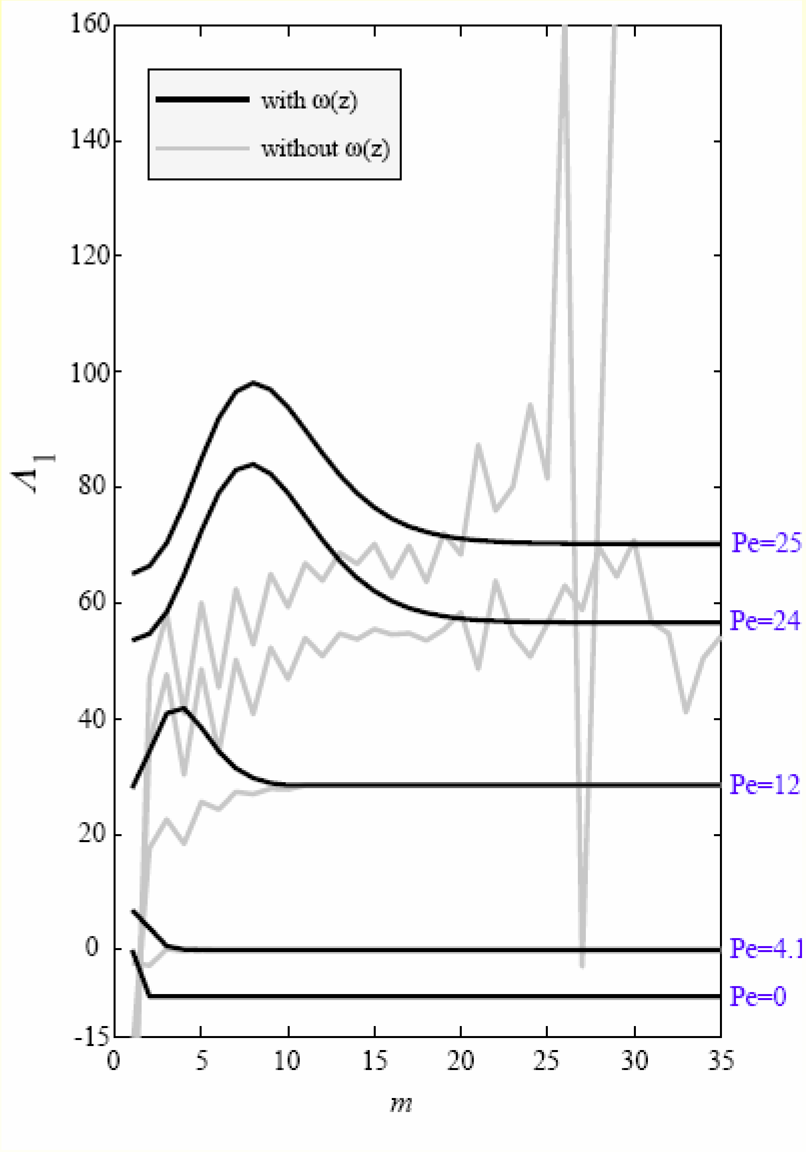
- Dynamics Behaviour:

  ➡ Use the low-order dimensional model to study the dynamic behaviour in time, starting from an arbitrary initial condition. Compare with full-model results.

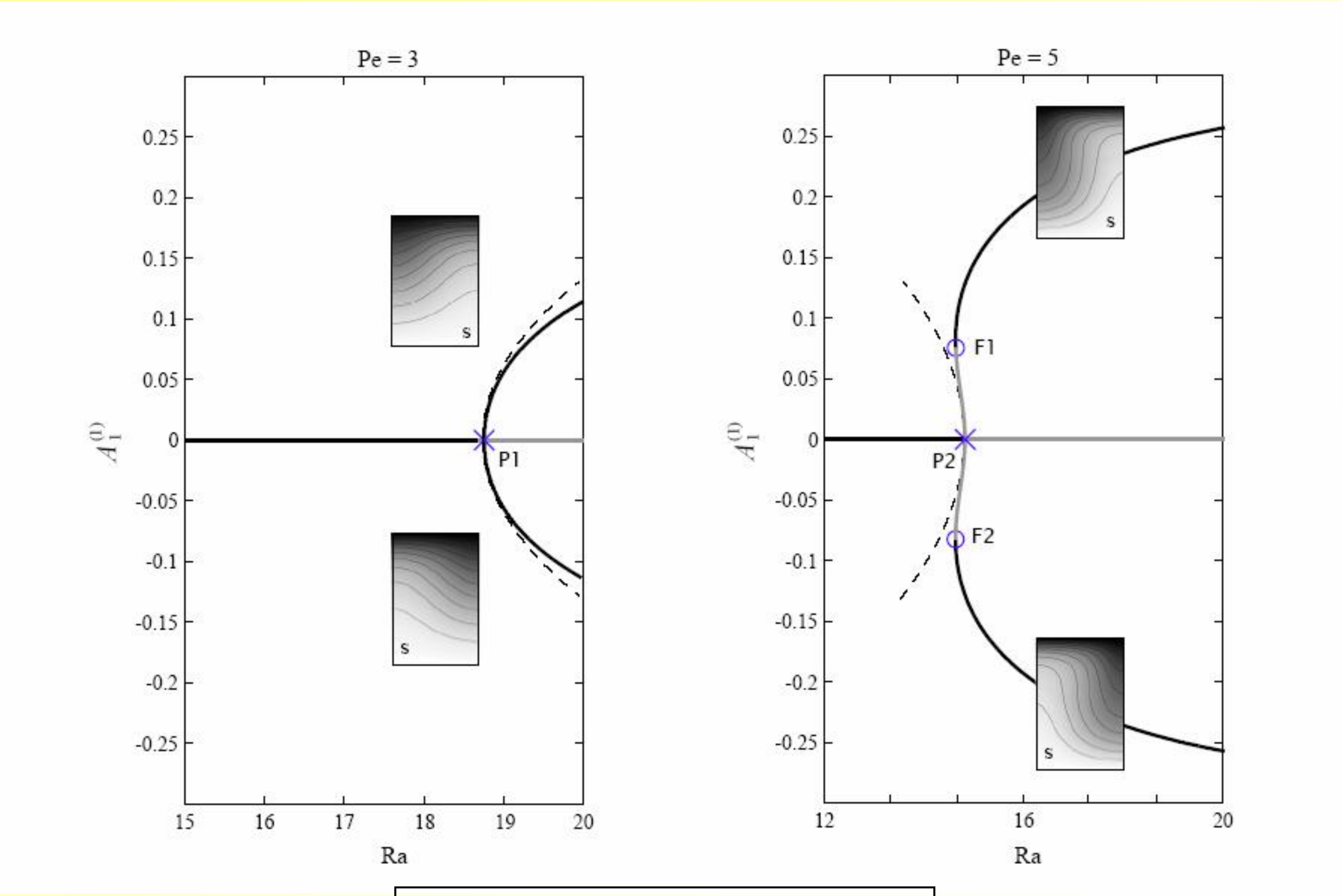
Bifurcation diagram close to critical R (1)

Landau Coefficient

- ➢Dependence on 'projection method'
- ➢Dependence on 'number of patterns'



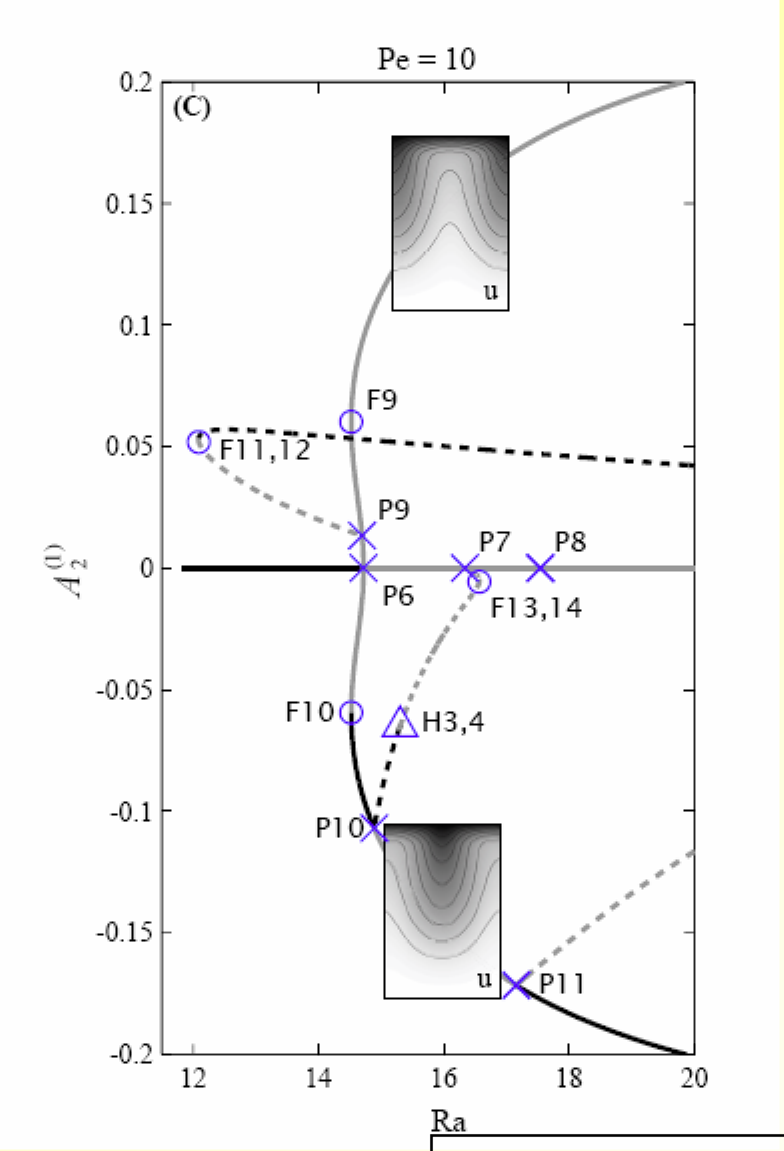
Bifurcation diagram for moderate R (1)



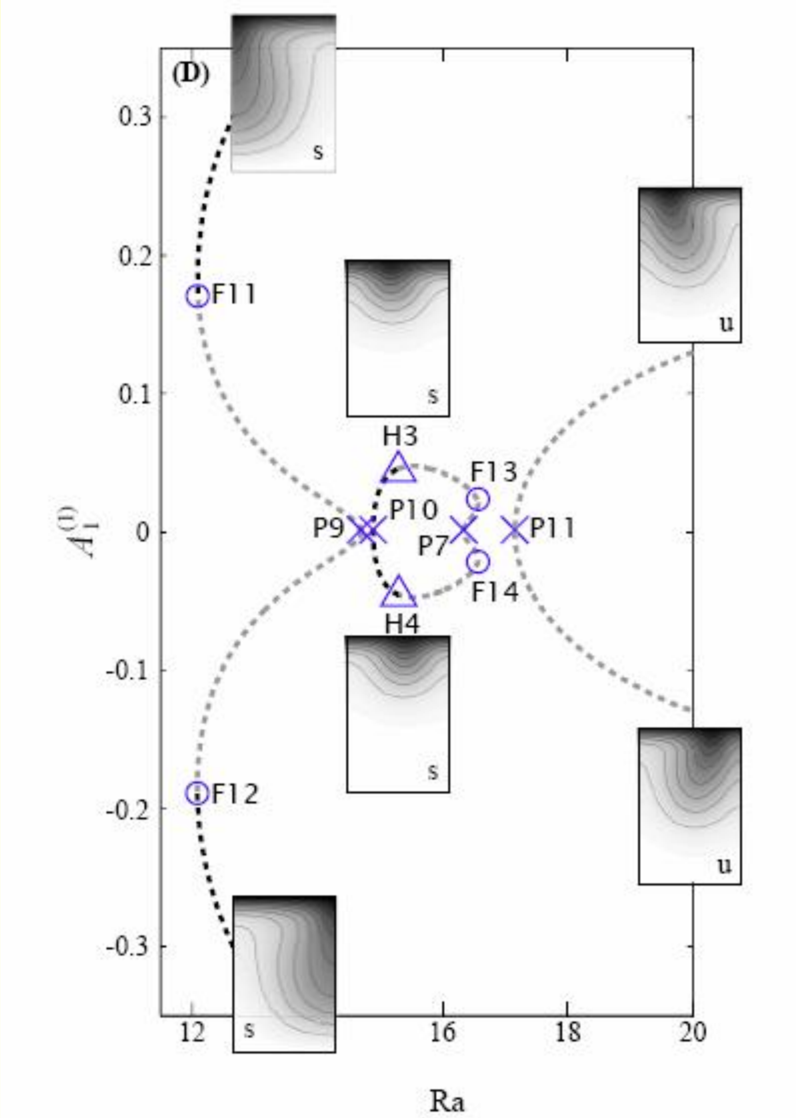
modes: n=10, m=10

Bifurcation diagram for moderate R (2)

Most unstable mode



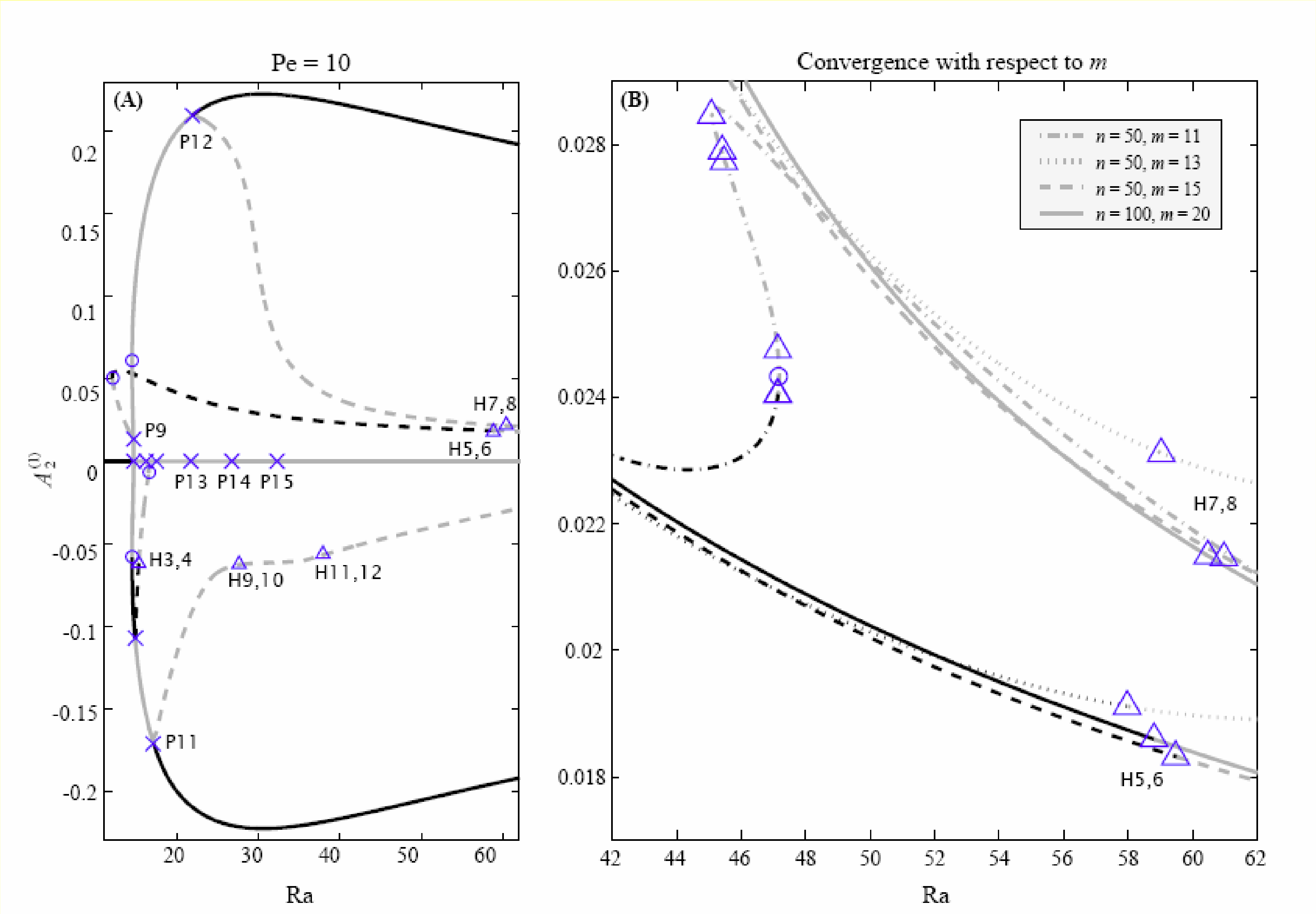
Slaved mode



modes: n=10, m=10

Bifurcation diagram for large R (1)

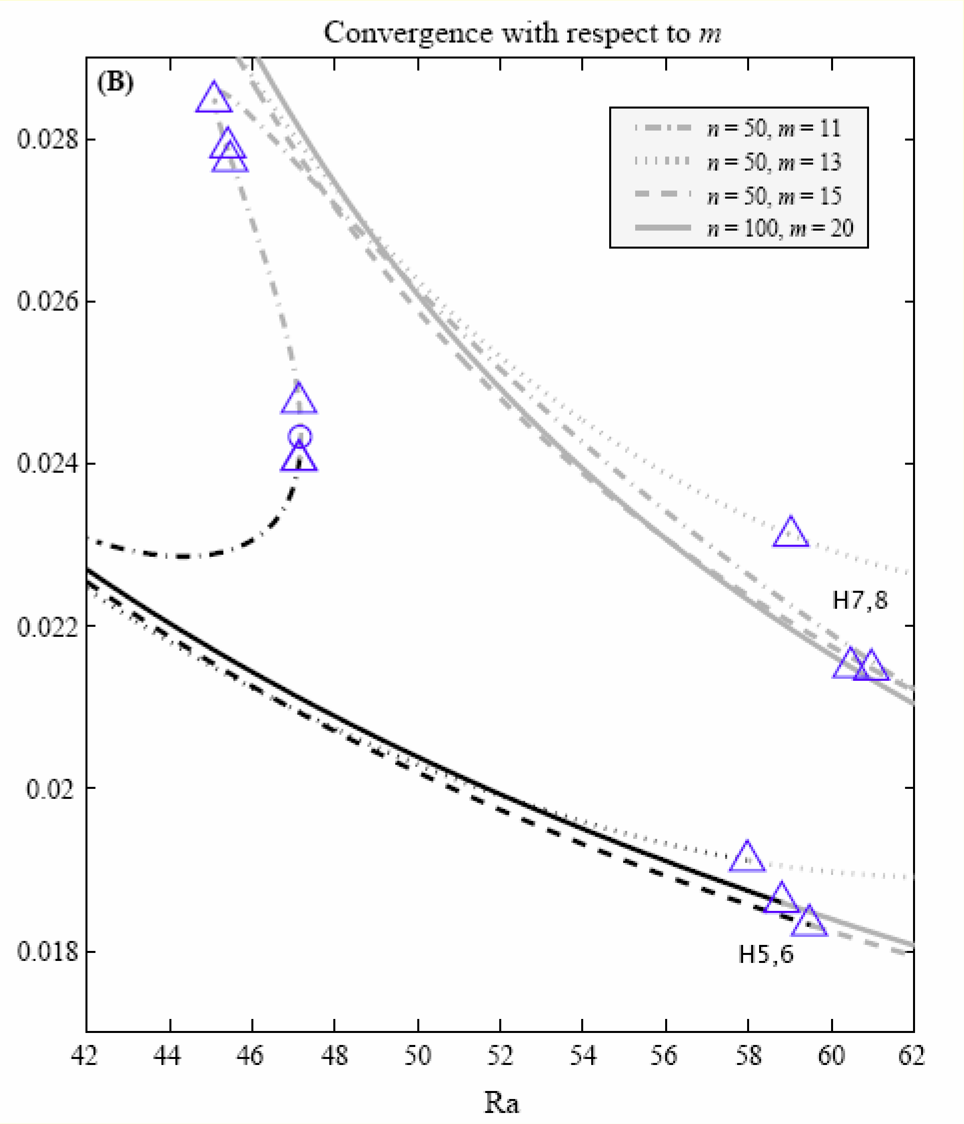
Most unstable mode



modes: n=100, m=20

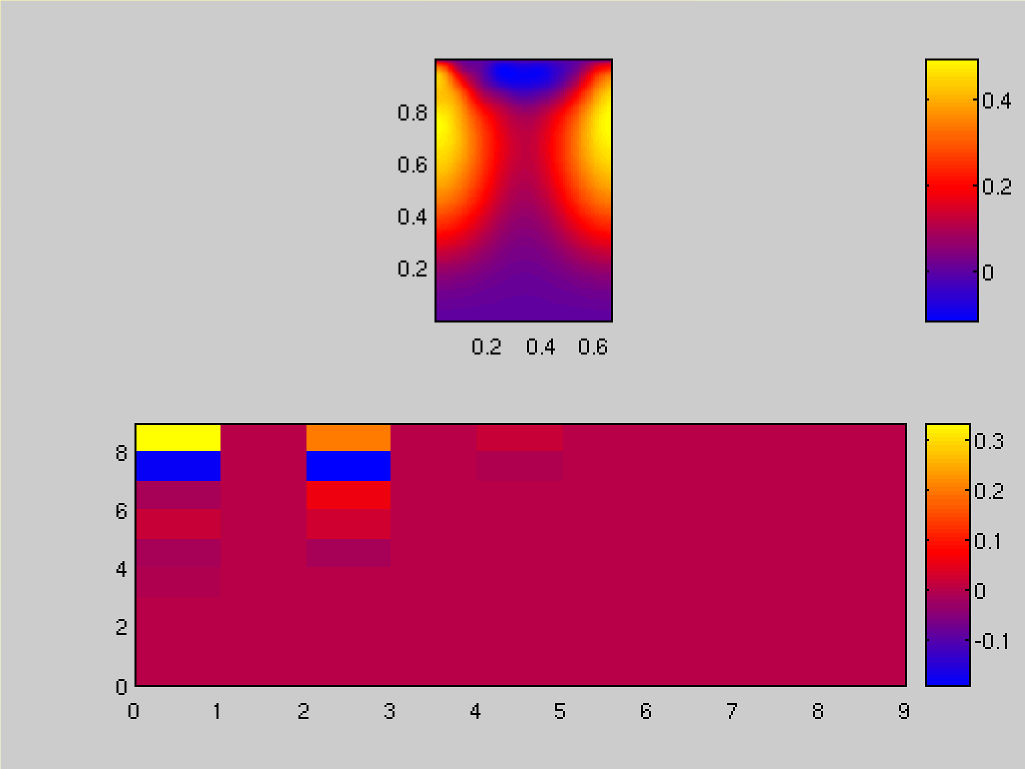
Bifurcation diagram for large R (2)

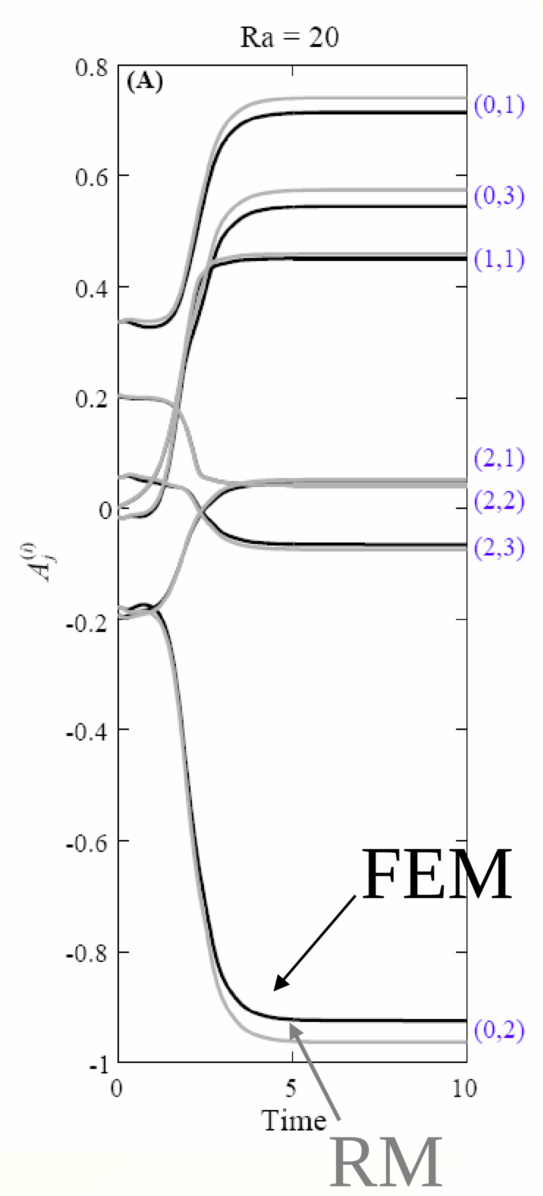
- Convergence:
  increase # of modes
  - z-modes: varied
  - x-modes: 100
- Sensitivity of
  bifurcation points
  to number of modes



Time evolution (1)

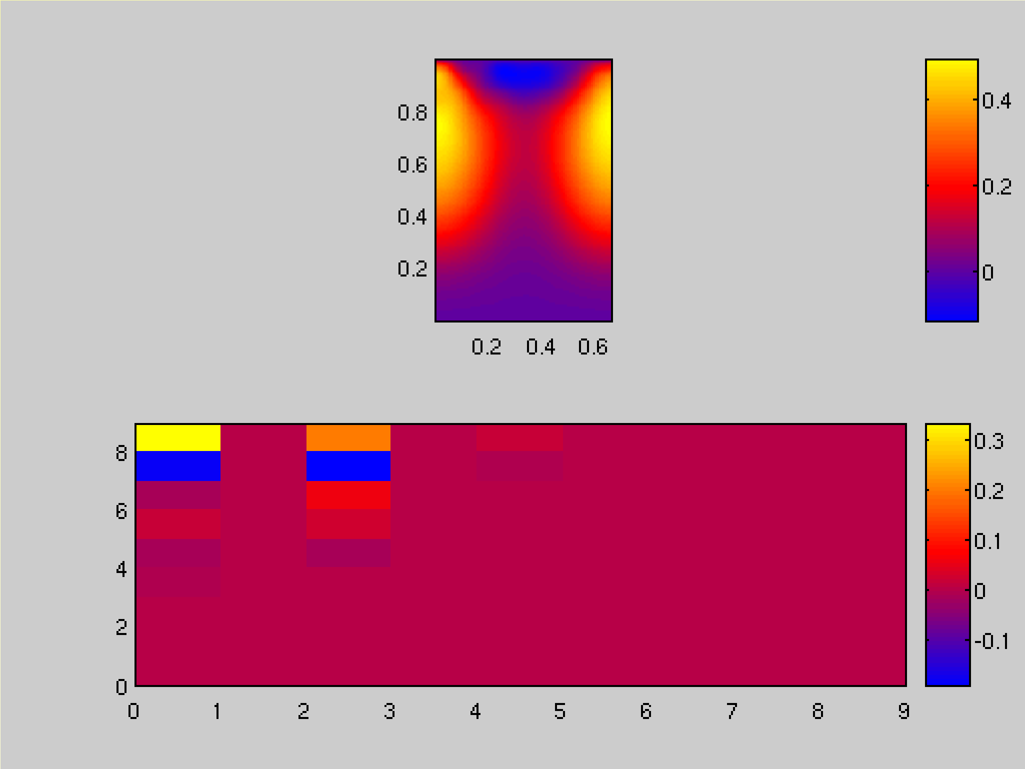
- Pe = 10, Ra = 20
- Initial condition: one-finger solution (lineaire most unstable mode)

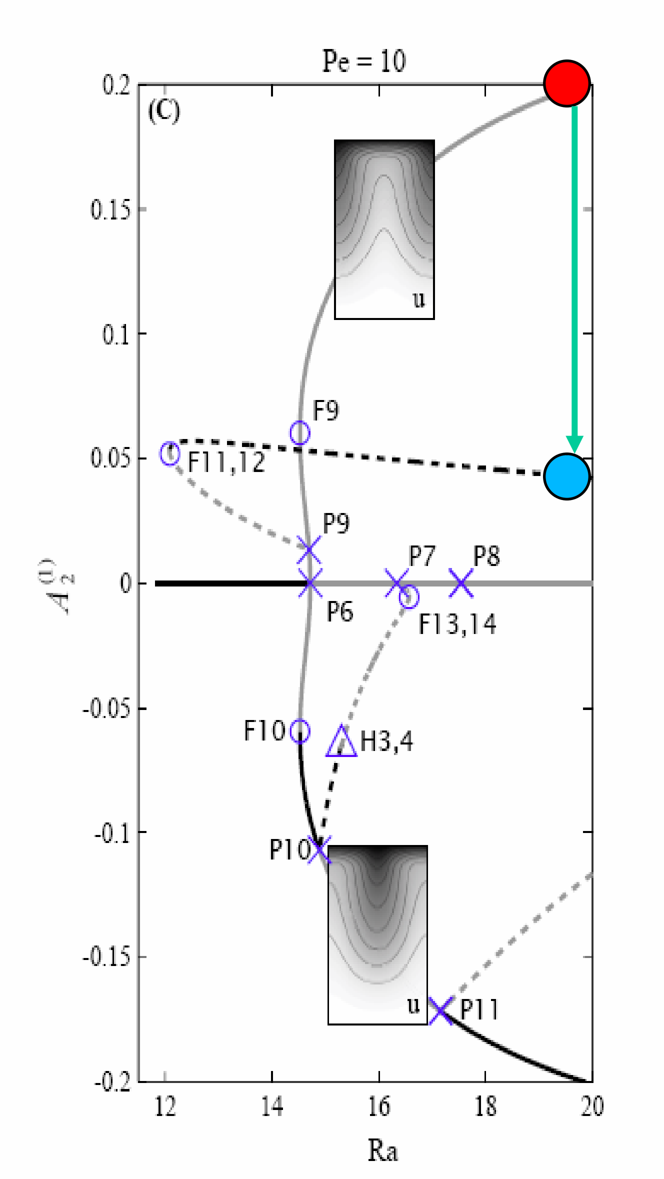




Time evolution (2)

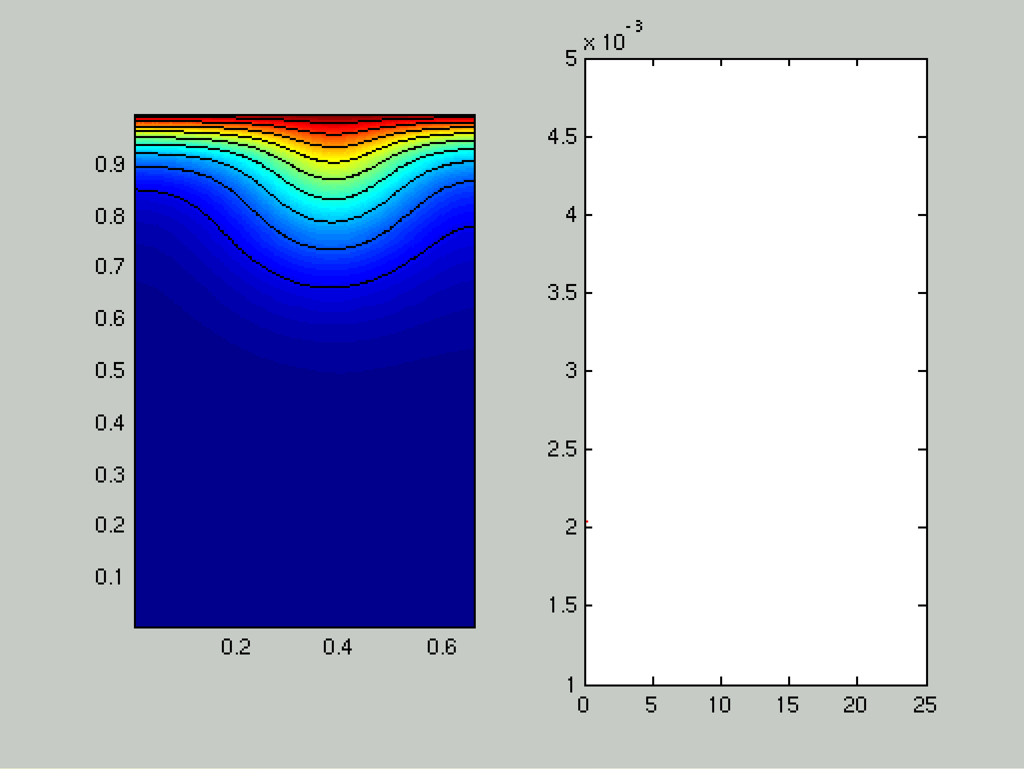
- Pe = 10, Ra = 20
- Initial condition: one-finger solution (lineaire most unstable mode)

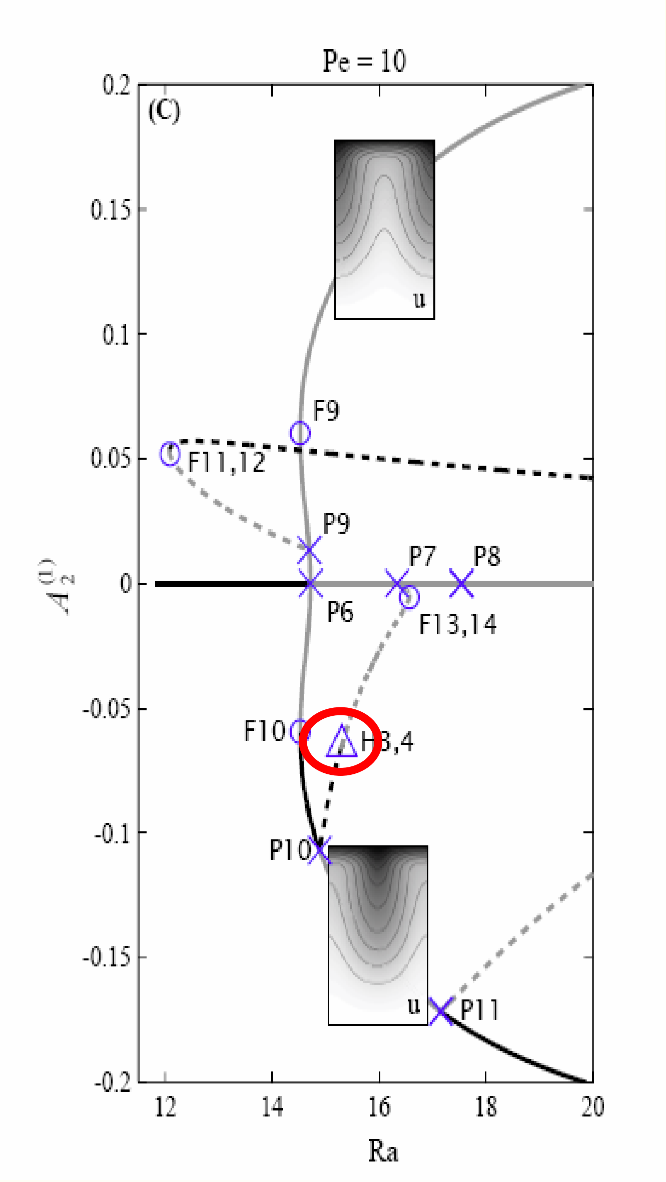




Time evolution (3)

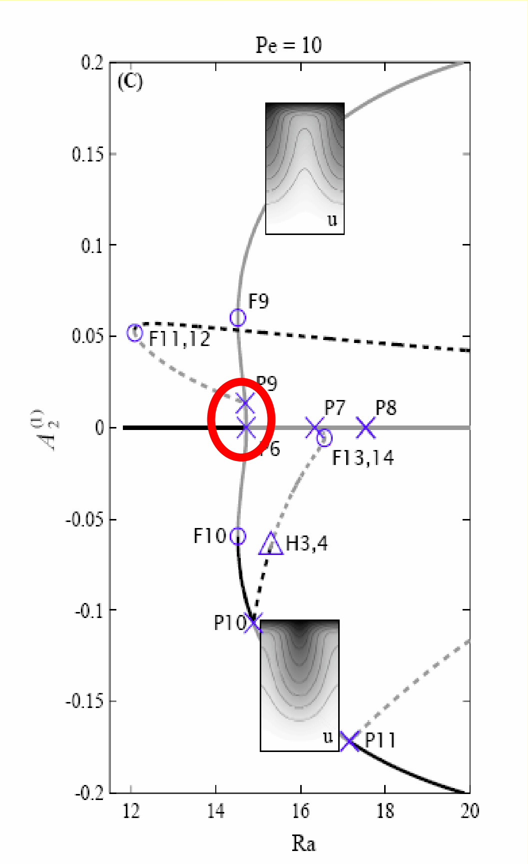
- Pe = 10, Ra = 15.35
- Initial condition: close to a Hopf

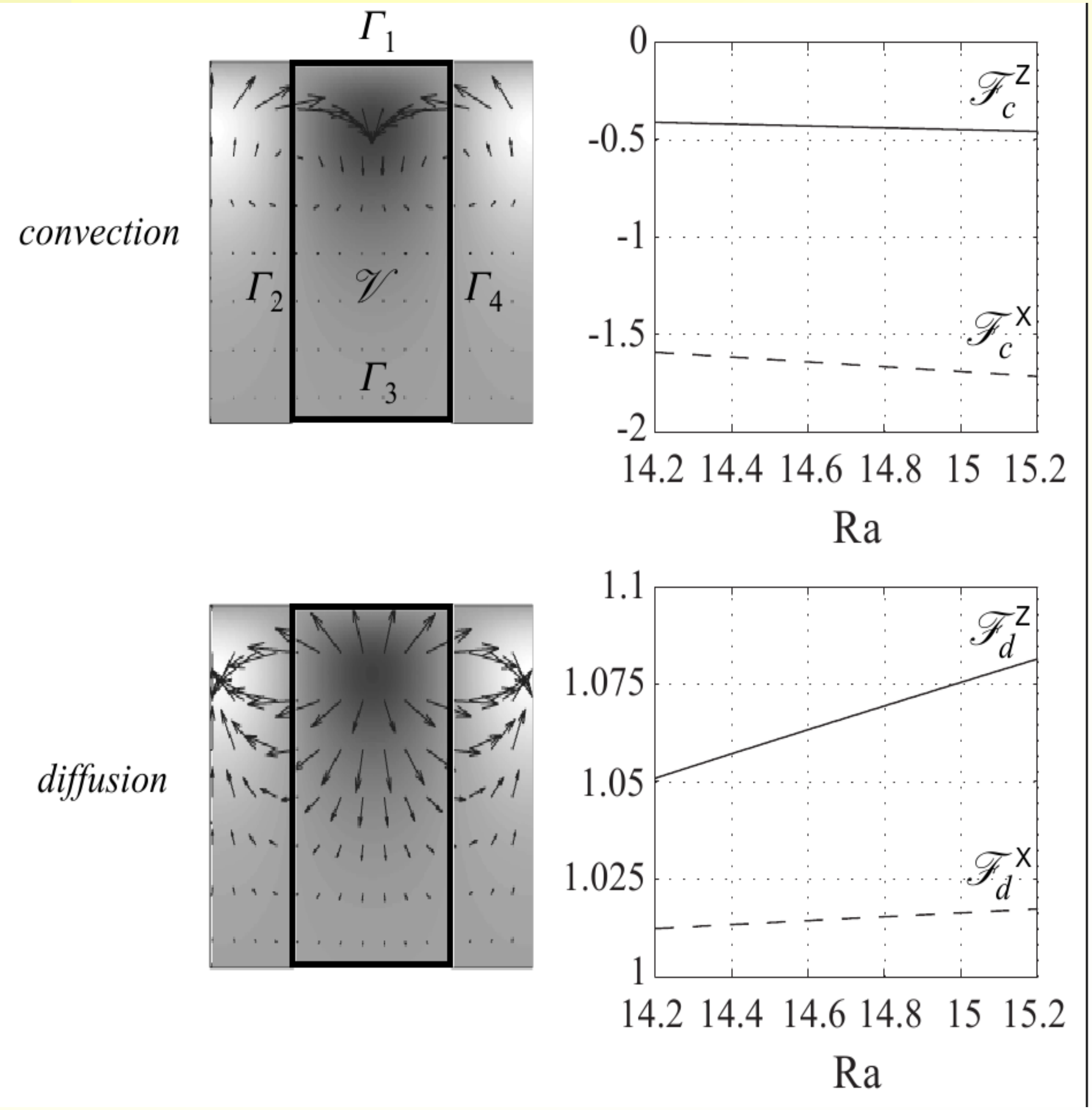




Mechanism (1)

Why gets the uniform solution unstable?

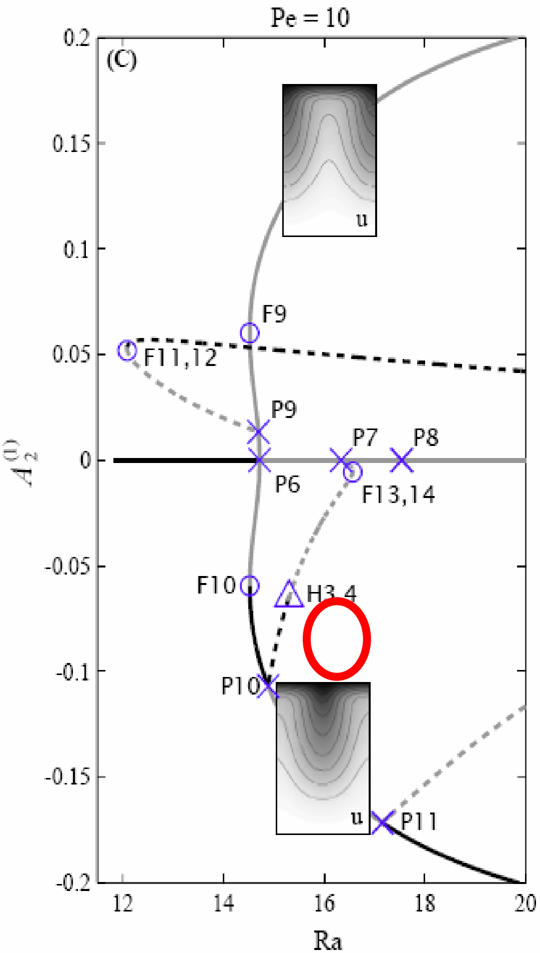


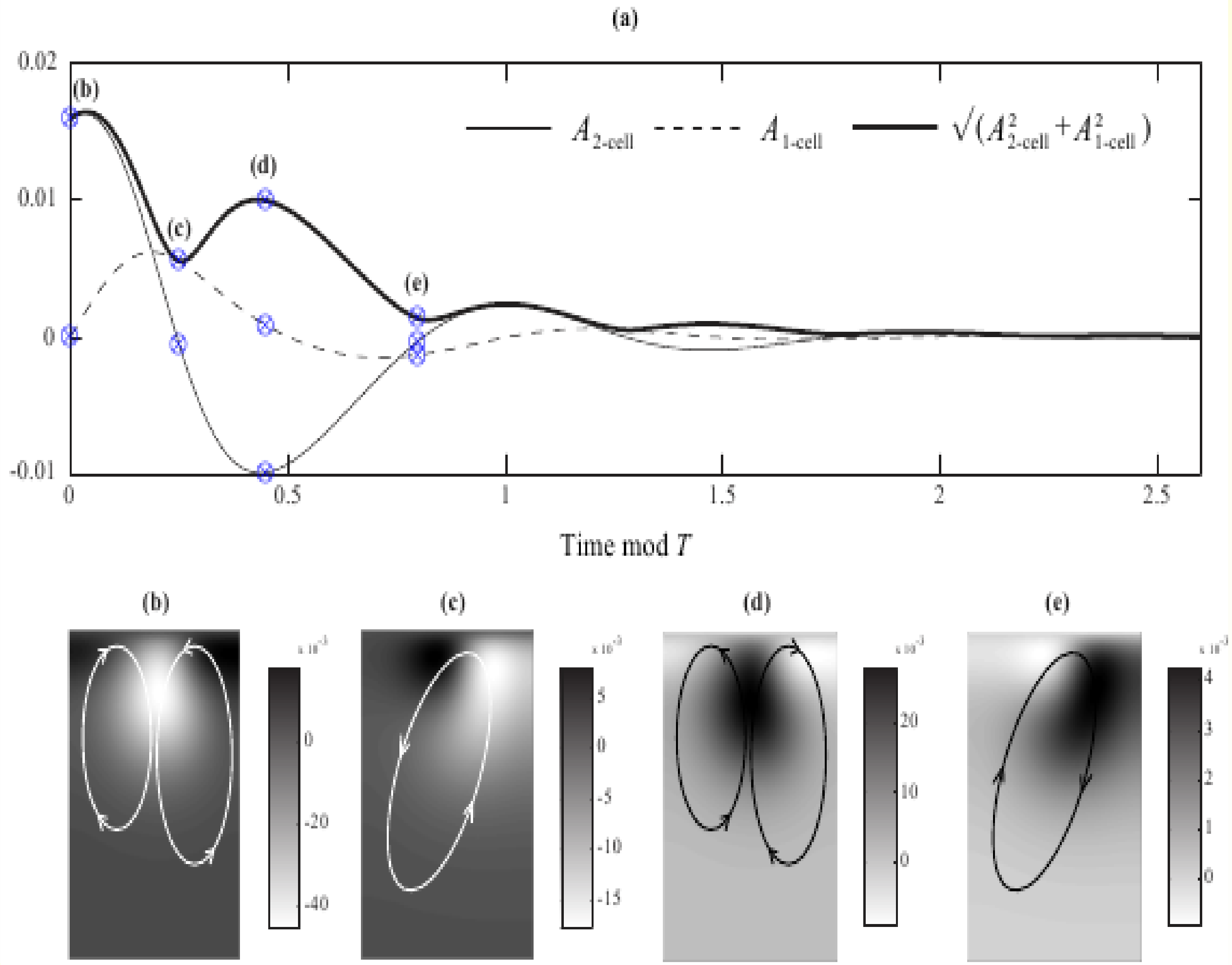


Mechanism (2)

Why a periodic solution?

Still stable….

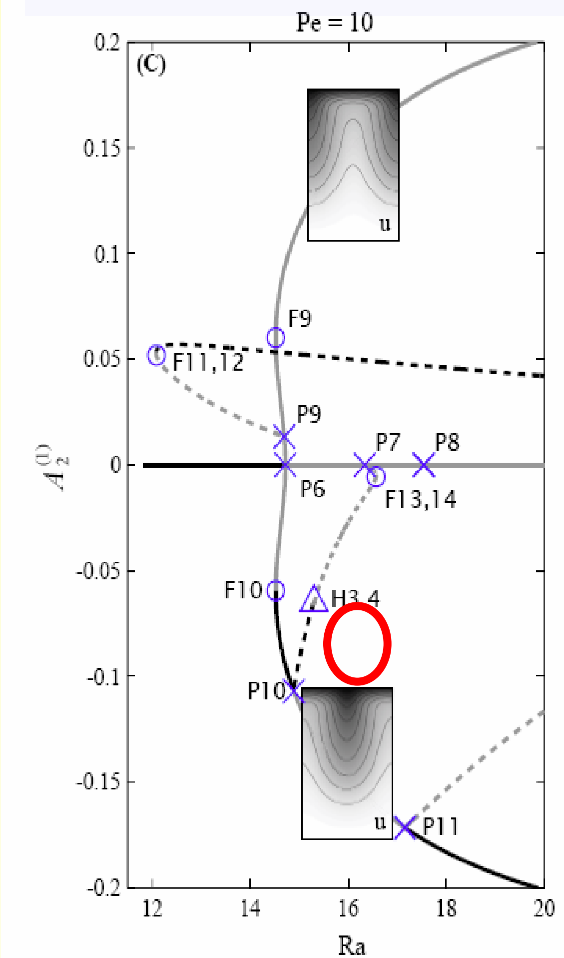


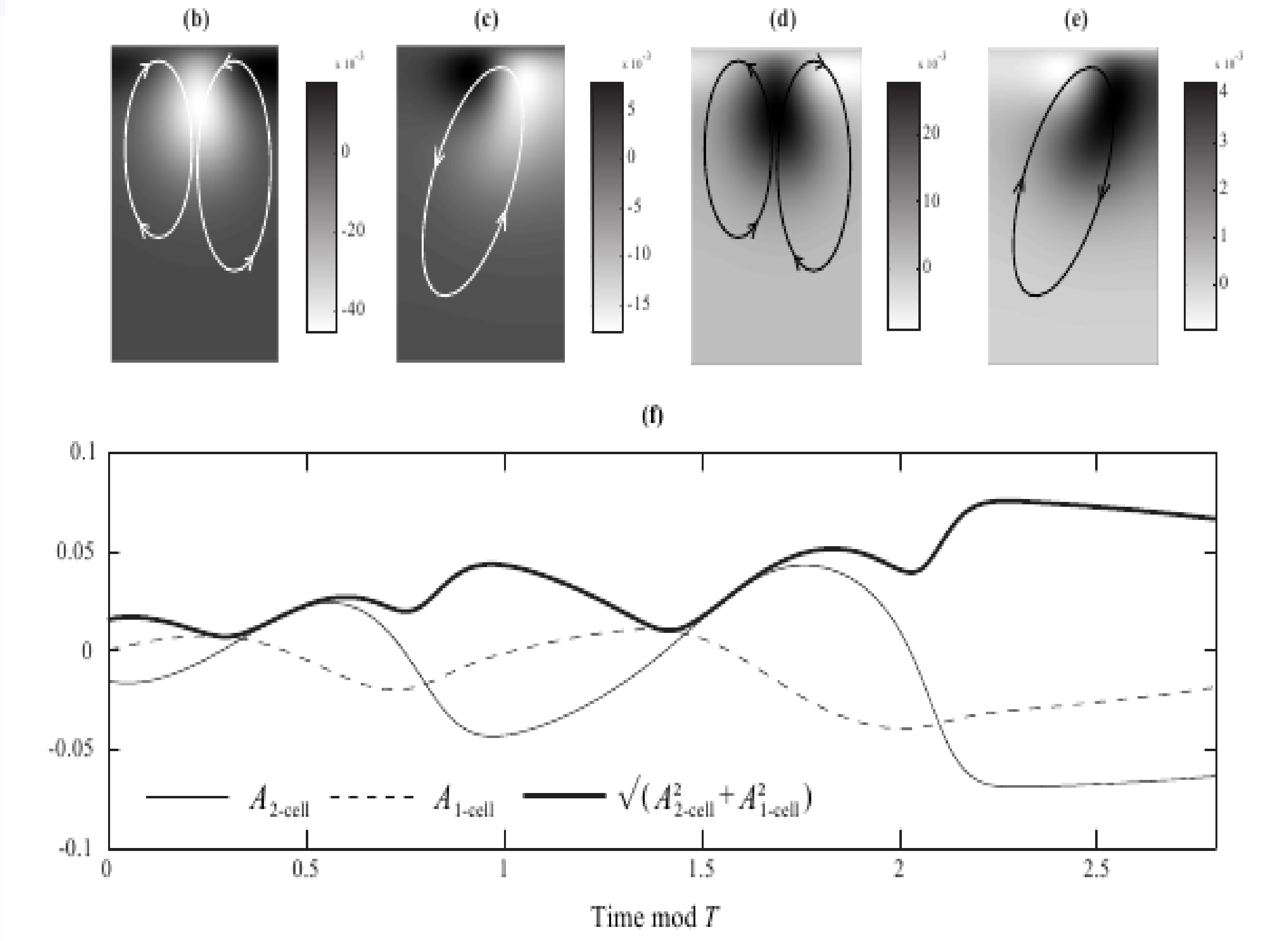


Mechanism (2)

Why a periodic solution?

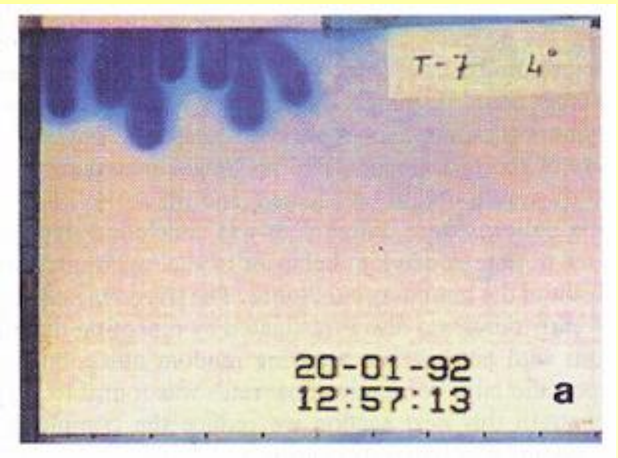
Now unstable….

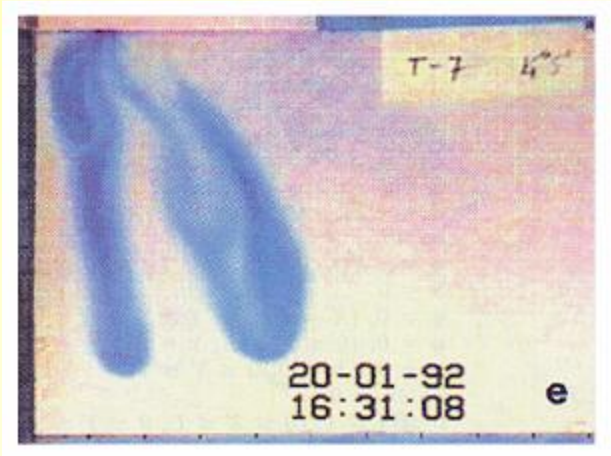


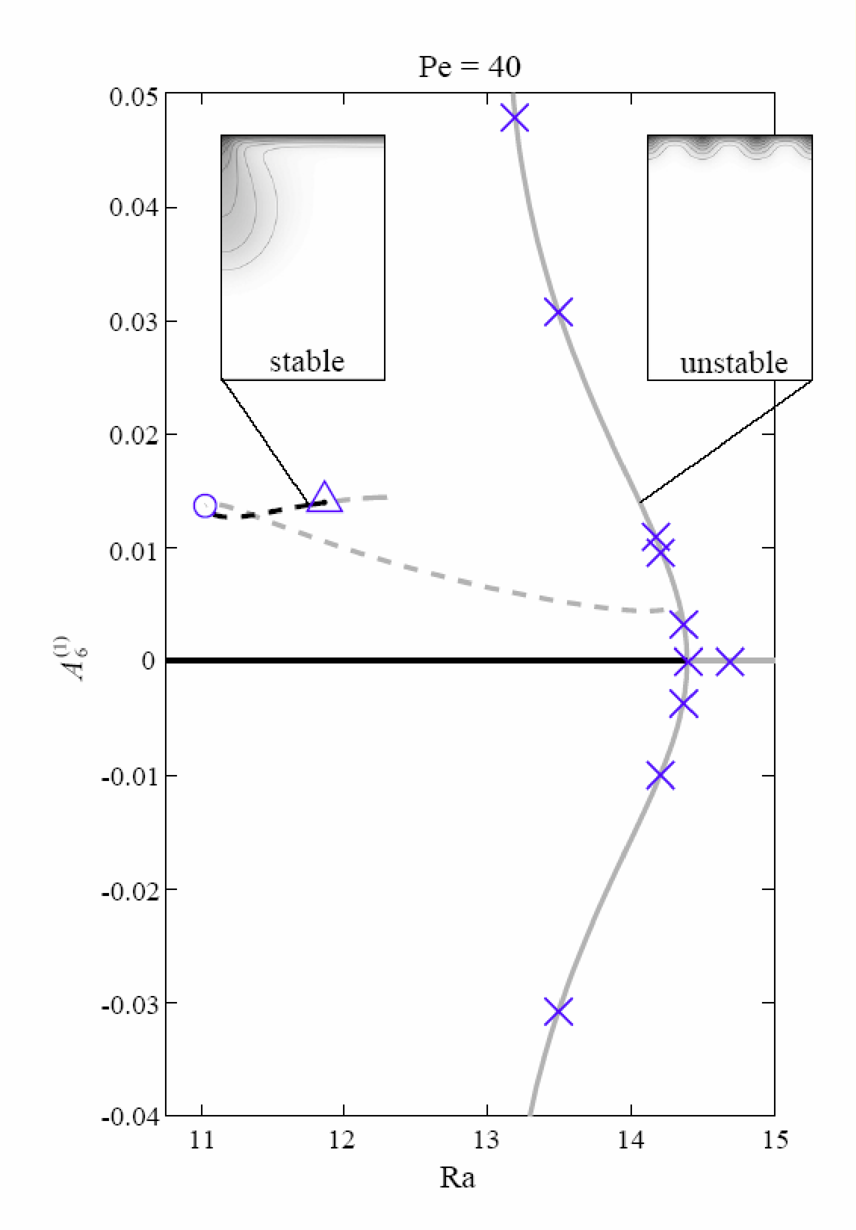


Comparison with observations (large Pe)

- Pe = 40, Ra varied
- n=30, m = 35







Conclusions

- Reduced model approach efficient in finding bifurcation structure in Salt Lake problem.
  - Convergence up to R~70 for Pe < 10, solutions recovered using FE simulations
  - Linearly most unstable mode does not necessarily predict observed length scales correctly (see Pe=10, Pe=40)
  - Multiple equilibria
  - Periodic solutions exist
- The low-dimensional dynamical model captures the dynamics of the full system of equations
- For larger Rayleigh numbers the basis obtained for R ~ 15 is not optimal anymore.

Conclusions (2)

- Method can be extended to 3 dimensions: