# A Singular Perturbation Approach in Nonlinear Aeroelasticity for Limit-Cycle Oscillations

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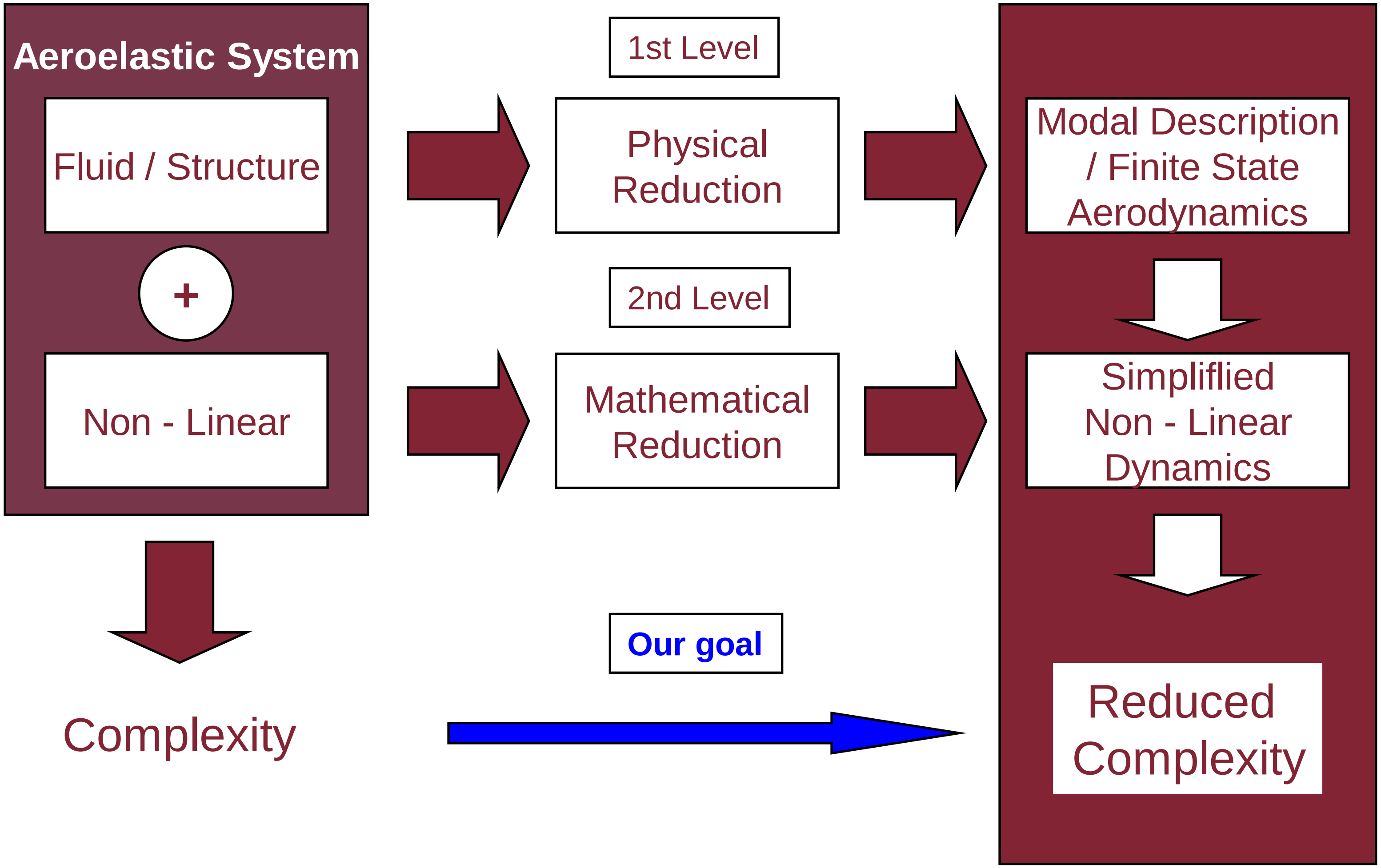
*Bifurcation and Model Reduction Techniques for Large Multi-Disciplinary Systems*

University of Liverpool 26-27 June 2008

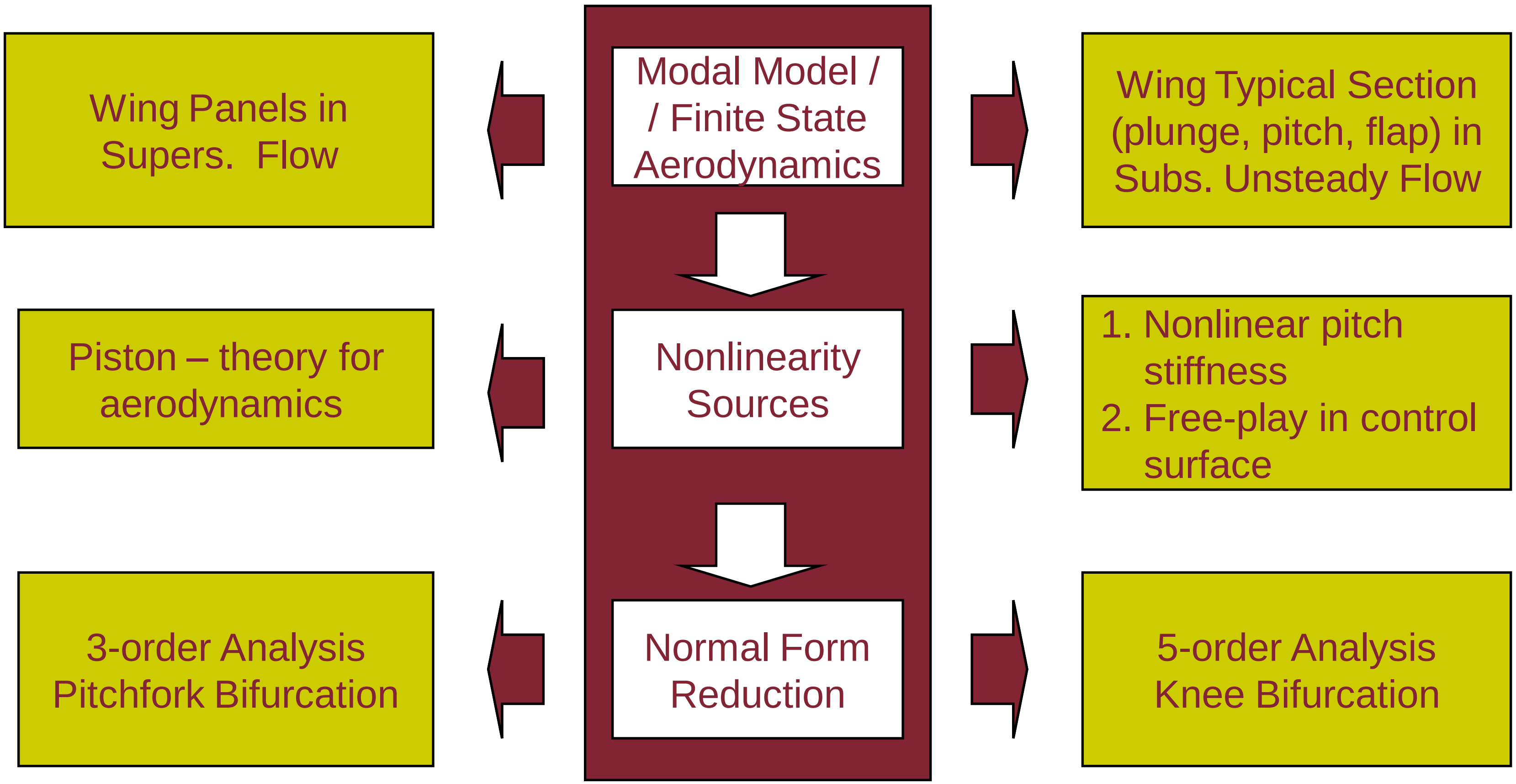
# Summary of presentation

1. Normal Form (NF) approach for nonlinear (NL) (**aeroelastic**) systems experiencing bifurcations: **highlights of theory for (aeroelastic) model reduction**
2. **NL analysis based on 3rd order** NF theory
   - Appl.#1: LCO's control
   - Appl.#2: Chaos control
3. **NL analysis based on 5th order** NF theory
   - Appl.#3: Gust response of a nonlinear aeroelastic system
   - Appl.#4: Freeplay modeling: direct num. integration v.s. NF approach
4. Concluding remarks

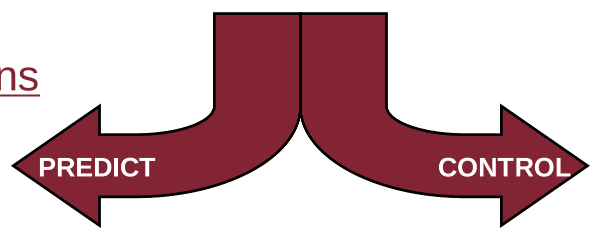
# Topics at a glance



# Considered aeroelastic applications and relevance of NF




<u>Prevision of safe conditions</u>
- flutter speed limits
- admissible gust




<u>Phenomenon intensity</u>
- LCO amplitude
- Chaos behaviour

# Authors’ papers related to the presentation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- Mastroddi, F., Bettoli, A., “Wavelet Analysis for Hopf Bifurcations with Aeroelastic Applications,” *Journal of Sound and Vibration*, 1999, **225**(5), pp. 887-913.
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- Carcaterra, A., Dessi, d., Mastroddi, “Hydrofoil Vibration Induced by a Random Flow: a Stochastic Perturbation Approach,” *Journal of Sound and Vibration*, 2005, **283**, pp. 401-432.
- Dessi, D., Mastroddi, F., Morino, L. “Normal-Form Analysis of Hopf Bifurcation Beyond the Center-Manifold Approximation”, proceedings of the *Euromech colloquium 457 on non linear modes of vibrating systems,* Frejus, France, June 7-9 2004.
- Dessi, D., Mastroddi, F., “Limit-Cycle Staibility Reversal via Singular Perturbation and Wing-Flap Flutter”, *Journal of Fluids and Structures,* Vol. 19, 2004, pp. 765-783.
- Dessi, D., Morino, L., Mastroddi, F., “A Fifth-Order Multiple-Scale Solution for Hopf Bifurcations,” *Computers and Structures,* Vol. 82, 2004, pp. 2723-2731.

## Some theoretical hightlights on

## NORMAL FORM (NF) approach for bifurcation

***-A tool for analysis of nonlinear (aeroelastic) systems with polynomial nonlinearities***

**Ali H. Nayfeh, Method of Normal Forms, Wiley Series in Nonlinear Science**

# Normal Form (I)

Control parameter (e.g., flight speed)

- Nonlinear Aeroelastic Problem **reduced** to : $\frac{d\boldsymbol{\xi}}{dt} = \mathbf{g}(\boldsymbol{\xi}, \mu)$
- Taking a Taylor expansion of $\mathbf{g}$ around an **equilibrium point** the dynamical system can be rewritten as:

$$\frac{d\mathbf{x}}{dt} = \hat{\mathbf{A}}(\mu)\mathbf{x} + \mathbf{f}(\mathbf{x}, \mu)$$

$\mathbf{f}$ nonlinear-term vector of type (generic NL's can be locally reduced in this form) :

$$f_n = \sum_{p,q} \hat{b}_{npq}(\mu) x_p x_q + \sum_{p,q,r} \hat{c}_{npqr}(\mu) x_p x_q x_r + \sum_{p,q,r} \hat{e}_{npqrst}(\mu) x_p x_q x_r x_s x_t + \dots .$$

- Assuming $\hat{\mathbf{A}}$ **analytically dependent on** $\mu$ one obtains

$$\hat{\mathbf{A}} = \hat{\mathbf{A}}|_{\mu=0} + \left.\frac{\partial \hat{\mathbf{A}}}{\partial \mu}\right|_{\mu=0} \mu + \left.\frac{\partial^2 \hat{\mathbf{A}}}{\partial \mu^2}\right|_{\mu=0} \mu^2 + \dots = \hat{\mathbf{A}}_0 + \mu \hat{\mathbf{A}}_2 + \mu^2 \hat{\mathbf{A}}_4 + \dots$$

Similarily:

$$\hat{\mathbf{c}} = \hat{\mathbf{c}}^0 + \mu \hat{\mathbf{c}}^1 + \mu^2 \hat{\mathbf{c}}^2 + \dots$$
$$\hat{\mathbf{e}} = \hat{\mathbf{e}}^0 + \mu \hat{\mathbf{e}}^1 + \mu^2 \hat{\mathbf{e}}^2 + \dots$$

# Normal Form (II)

- Setting the transformation $\mathbf{x} = \mathbf{R}\mathbf{z}$ (**diagonalize the linear-ized part**), one has ( $\mathbf{\Lambda}$ diagonal matrix of eigenvalues $\lambda_1, \lambda_2, \ldots$ ):

$$\dot{\mathbf{z}} = (\mathbf{\Lambda} + \mu \mathbf{A}_2)\mathbf{z} + \sum_{p,q,r} \gamma_{npqr} z_p z_q z_r + \sum_{p,q,r,s,t} \zeta_{npqrst} z_p z_q z_r z_s z_t + \mu^2 \mathbf{A}_4 \mathbf{z}$$

$$+\mu \sum_{p,q,r} \delta_{npqr} z_p z_q z_r + O(\|\mathbf{z}^5\|) + O(\mu^2) + O(\mu^2 \|\mathbf{z}^3\|) + O(\mu \|\mathbf{z}^5\|)$$

where

$$\delta_{npqr} = \textstyle\sum_{p,q,r} R^{-1}_{js} c^1_{stuv} R_{tp} R_{uq} R_{vr}, \ \mathbf{A}_2 = \mathbf{R}^{-1} \hat{\mathbf{A}}_2 \mathbf{R}, \ \mathbf{A}_1 = \mathbf{R}^{-1} \hat{\mathbf{A}}_1 \mathbf{R}$$

$$\zeta_{npqrst} = \textstyle\sum_{p,q,r,s,t} R^{-1}_{ju} e^0_{uvxwyz} R_{vp} R_{xq} R_{wr} R_{ys} R_{zt}, \text{ and } \gamma_{npqr} = \textstyle\sum_{p,q,r} R^{-1}_{js} c^0_{stuv} R_{tp} R_{uq} R_{vr}$$

- **The ordering parameter** $\epsilon$ is introduced such that $\mathbf{z} = \epsilon \mathbf{u}$
Motivation: scaling the contributions of any terms in each equations on the base of the amplitude of the original state space variable $\mathbf{z}$

- Balancing the nonlinear terms with the perturbation on the linear terms $\mu \mathbf{A}_2 \mathbf{u}$ (condition for a local bifurcation) implies →

$$\mu = \mu_2 \epsilon^2 + \mu_4 \epsilon^4$$

## Normal Form (III)

- Recasting equations in the new state space variable $\mathbf{u}$

$$\dot{\mathbf{u}} = \boldsymbol{\Lambda}\mathbf{u} + \epsilon^2[\check{\mathbf{A}}_2\mathbf{u} + \sum_{p,q,r} \gamma_{npqr}(\mu) z_p z_q z_r]$$
$$\epsilon^4[\check{\mathbf{A}}_4\mathbf{u} + \sum_{p,q,r} \delta_{npqr} u_p u_q u_r + \sum_{p,q,r,s,t} \zeta_{npqrst} u_p u_q u_r u_s u_t]$$

$$\check{\mathbf{A}}_4 = \mu_2^2 \mathbf{A}_4 + \mu_4 \mathbf{A}_2, \; \check{\mathbf{A}}_2 = \mu_2 \mathbf{A}_2$$

or

$$\dot{\mathbf{u}} = \boldsymbol{\Lambda}\mathbf{u} + \epsilon^2 \mathbf{f}^{(2)}(\mathbf{u}) + \epsilon^4 \mathbf{f}^{(4)}(\mathbf{u})$$

- The normal form method consists of **simplifying the differential problem** through the “near identity” coordinate transformation

$$\mathbf{u} = \mathbf{y} + \epsilon^2 \mathbf{w}^{(2)}(\mathbf{y}) + O(\epsilon^4)$$

where $\mathbf{w}^{(2)}(\mathbf{y})$ have to be chosen so as to simplify the problem.

# Normal Form (IV)

- **Substituting** ($\mathbf{W} := d\mathbf{w}^{(2)}/d\mathbf{y}$):

$$\dot{\mathbf{u}} = \dot{\mathbf{y}} + \epsilon^2 \mathbf{W}(\mathbf{y})\dot{\mathbf{y}} + O(\epsilon^4) = \mathbf{\Lambda}(\mathbf{y} + \epsilon^2 \mathbf{w}^{(2)}) + \epsilon^2 \mathbf{f}^{(2)}(\mathbf{y} + \epsilon^2 \mathbf{w}^{(2)}) + O(\epsilon^4)$$

- **Collecting** terms of same order

$$\dot{\mathbf{y}} = \mathbf{\Lambda y} + \epsilon^2 \mathbf{g}^{(2)}(\mathbf{y}) + O(\epsilon^4)$$

where

$$\mathbf{g}^{(2)} = -\mathbf{W\Lambda y} + \mathbf{\Lambda w}^{(2)}(\mathbf{y}) + \mathbf{f}^{(2)}(\mathbf{y})$$

NEW nl term — DIFFERENCE ($-\mathbf{W\Lambda y} + \mathbf{\Lambda w}^{(2)}(\mathbf{y})$) — OLD nl term ($\mathbf{f}^{(2)}(\mathbf{y})$)

- **Choosing** for $\mathbf{w}^{(2)}(\mathbf{y})$ the same functional dependence as $\mathbf{f}^{(2)}(\mathbf{y})$

$$w_n^{(2)}(y_k) = \sum \alpha_{np} y_p + \sum \Gamma_{npqr} y_p y_q y_r$$

- The following expressions are obtained for coefficients of $\mathbf{w}^{(2)}(\mathbf{y})$

$$\alpha_{np} = \begin{cases} \frac{a_{np}}{\lambda_p - \lambda_n} & \text{otherwise} \\ 0 & \text{if } \| \lambda_p - \lambda_n \| \leq \rho \end{cases}$$

$$\Gamma_{npqr} = \begin{cases} \frac{\gamma_{npqr}}{\lambda_p + \lambda_q + \lambda_r - \lambda_n} & \text{otherwise} \\ 0 & \text{if } \| \lambda_p + \lambda_q + \lambda_r - \lambda_n \| \leq \rho \end{cases}$$

**(near)-Resonance conditions**

**$\rho$ Suitable small positive parameter defining the resonance conditions**

# Normal Form (V) RELEVANT ISSUE #1: opportunity of defining the *near-identity* transformation

**Given system**

$$\hat{\lambda} := \lambda_{j_1} + \ldots + \lambda_{j_N} - \lambda_i$$

**Pseudo-diagonal system**

**Near-Resonance condition**

$$\left\| \hat{\lambda} \right\| \le \rho$$

$$\left\| \hat{\lambda} \right\| > \rho$$

**Normal form**

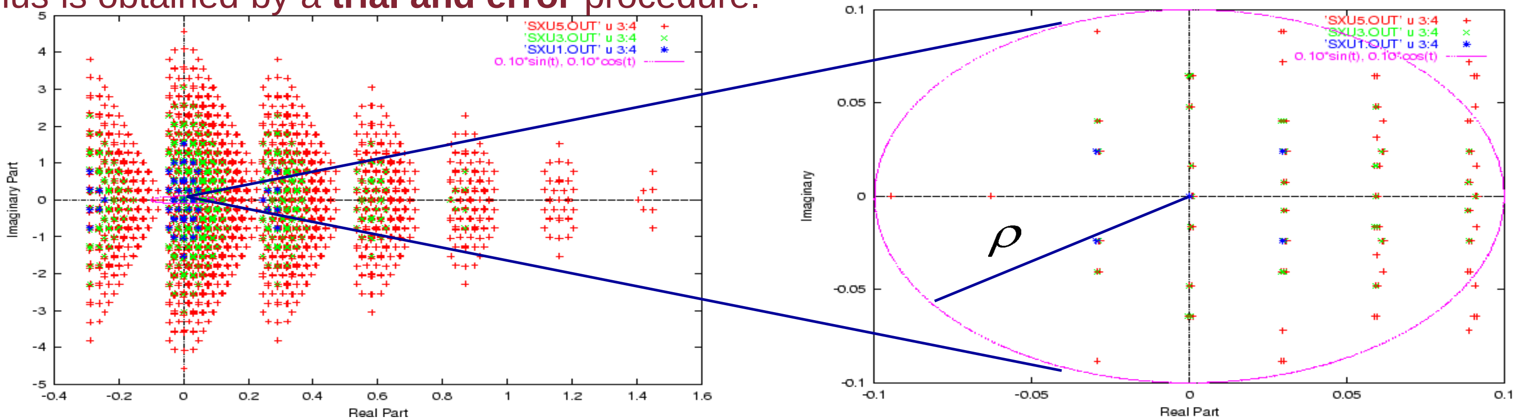
$$\dot{\mathbf{y}} = \Lambda \mathbf{y} + \mathbf{g}^{(2)}(\mathbf{y})$$

**Nonlinear transformation**

$$\mathbf{z} = \mathbf{y} + \mathbf{w}^{(2)}(\mathbf{y})$$

In the complex plane, the $\hat{\lambda}$ -points can be plotted for every nonlinear term (green for 3rd order terms, blue for linear terms); by **enlarging** the **circle** of radius $\rho$ **more nonlinear terms** are taken into account until a **satysfying solution** is reached.

The radius is obtained by a **trial and error** procedure.



# Normal Form (V) RELEVANT ISSUE #2: LCO solution analytically obtained (resonance conditions) $\rho = 0$

$\beta_R$ Directly related to the linear part

$\gamma_R$ Directly related to the lnoninear part

- Third order system

$$\dot{y}_n = \lambda_n y_n + \epsilon^2 \left( \sum_{\hat{\lambda}=0} a_{np} y_p + \sum_{\hat{\lambda}=0} \gamma_{npqr} y_p y_q y_r \right)$$

- Consider the pair of complex conjugate equations given by the "manifold"

$$\begin{aligned} \dot{y}_1 &= \lambda_1 y_1 + \epsilon^2 [a_{11} y_1 + (\gamma_{1121} + \gamma_{1211} + \gamma_{1112}) y_1^2 y_2] \\ \dot{y}_2 &= \lambda_2 y_2 + \epsilon^2 [a_{22} y_2 + (\gamma_{2122} + \gamma_{2212} + \gamma_{2221}) y_2^2 y_1] \end{aligned}$$

Around a complex conjugated stable Complex eigenvalue

- **ANALYTIC SOLUTION FOR LCO** $\left( y_1 = a_1 e^{j\phi_1} \quad y_2 = a_2 e^{j\phi_2} \right)$

$$a_1(t) = \left[ \frac{-\beta_R^{(1)} / \gamma_R^{(1)}}{1 + k e^{2\epsilon^2 \beta_R^{(1)} t}} \right]^{1/2}$$

$$\phi_1(t) = \lambda_{1_I} t + \epsilon^2 t (-\beta_I^{(1)} + \gamma_I^{(1)} \beta_R^{(1)} / \gamma_R^{(1)} + \gamma_I^{(1)} / \gamma_R^{(1)} \ln(a_1) + \phi_1^0$$

# Normal Form (VI) (the other equations are "slaves")

- Consider the other eigenvalues

Stable Complex eigenvalue

$$a_n = a_n^0 \left(\frac{a_1}{a_1^0}^{(n)}\right)^{\gamma_R^{(n)}/\gamma_R^{(1)}} e^{\epsilon^2 t(-\beta_R^{(n)}+\beta_R^{(1)}\gamma_R^{(n)}/\gamma_R^{(1)})}$$

$$\varphi_n = imag(\lambda_n)t + \epsilon^2 t(-\beta_I^{(n)} + \gamma_I^{(n)} c/\gamma_R^{(1)}) + (\gamma_I^{(n)}/\gamma_R^{(1)}) ln(a_1) + \varphi_n^0$$

Stable Real eigenvalue

$$y_m = y_m^0 \left(\frac{a_1}{a_1^0}\right)^{\gamma_R^{(m)}/\gamma_R^{(1)}} e^{t(-\beta_R^{(m)}/\beta_R^{(1)})} e^{t(-\beta_R^{(m)})+\beta_R^{(1)}\gamma_R^{(m)}/\gamma_R^{(1)}}$$

- **Final analytical Solutions**

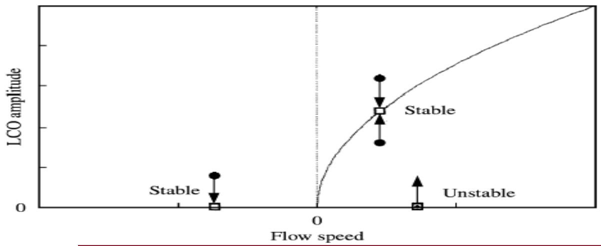
$$\mathbf{x} = \epsilon(\mathbf{r}^{(1)} a_1 e^{j\varphi_1} + \mathbf{r}^{(2)} a_1 e^{-j\varphi_1} + \sum_{n=3}^{N_c} \mathbf{r}^{(n)} a_n e^{-j\varphi_n} + \sum_{m=N_c+1}^{N} \mathbf{r}^{(m)} y_m)$$

# Normal Form (V) RELEVANT ISSUE #3: othe *stability of LCO* can be discussed (Hopf theorem)

**Case A:** $\gamma_R > 0$ **(nonlinear stabilizing terms)** **Case B:** $\gamma_R < 0$ **(nonlinear destabilizing terms)**

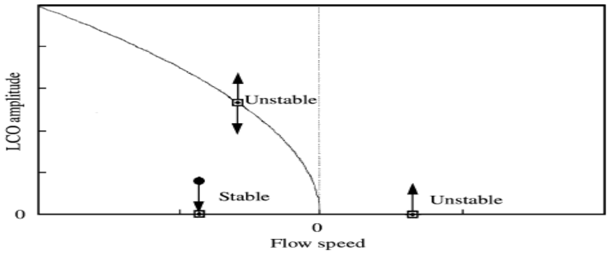
**Case A**

1. $\beta_R < 0$ (*i.e.*, $\mu > 0$, $\mu = +\epsilon^2$, linear destabilizing terms)
   - $a_1|_{t=0} > |\beta_R/\gamma_R|^{1/2} \Longrightarrow$ tends to a limit cycle from above
   - $a_1|_{t=0} < |\beta_R/\gamma_R|^{1/2} \Longrightarrow$ tends to a limit cycle from below
   - Hence, there is a stable limit cycle
2. $\beta_R > 0$ (*i.e.*, $\mu < 0$, $\mu = -\epsilon^2$, linear stabilizing terms)
   - Solution always tends to $a_1 = 0$




**Case B**

1. $\beta_R < 0$ (*i.e.*, $\mu > 0$, linear destabilizing terms) $a_1 \to \infty$ after infinite time
2. $\beta_R > 0$ (*i.e.*, $\mu < 0$, $\mu = -\epsilon^2$, linear stabilizing terms)
   - $a_1|_{t=0} > |\beta_R/\gamma_R|^{1/2} \Longrightarrow a_1 \to \infty$ after a finite time
   - $a_1|_{t=0} < |\beta_R/\gamma_R|^{1/2} \Longrightarrow a_1 \to 0$ for $t \to \infty$
   - Hence, there is an unstable limit cycle



# *Aeroelastic applications background*: linear case

Flow Speed

μ System parameter

Unsteady Load

Fluid (linear)

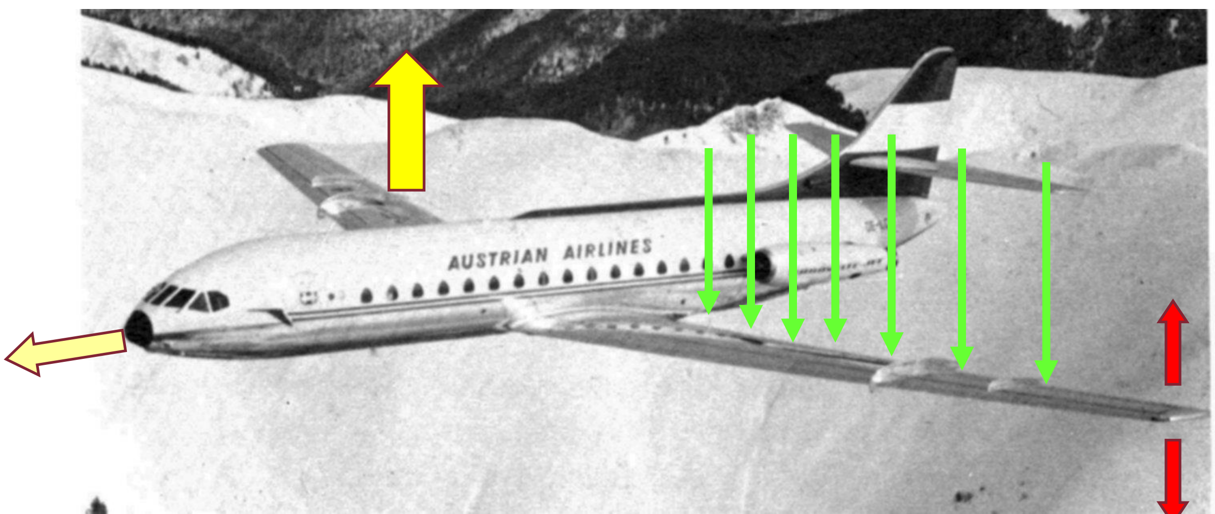
Gust Load

Structure (linear)

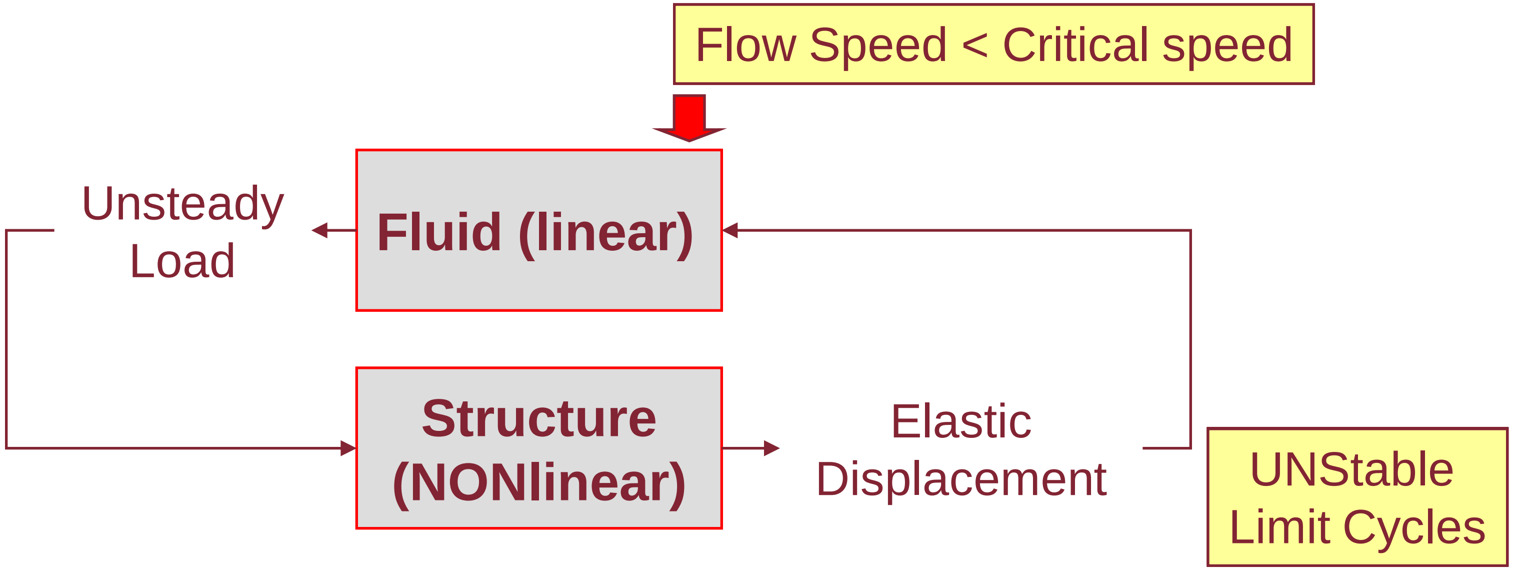
Elastic Displacement

Initial Values

Excitation mechanism

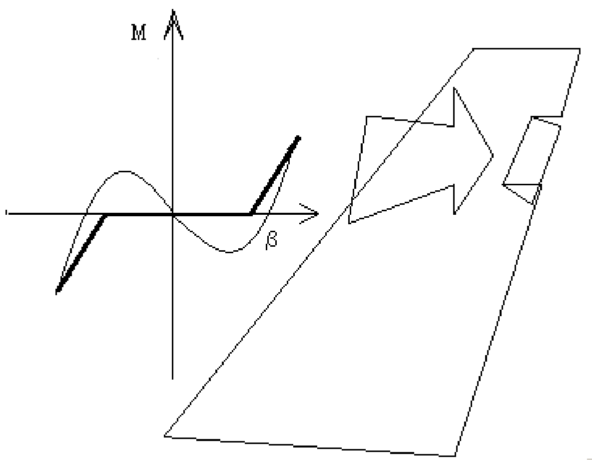


# Background: nonlinear case (1)

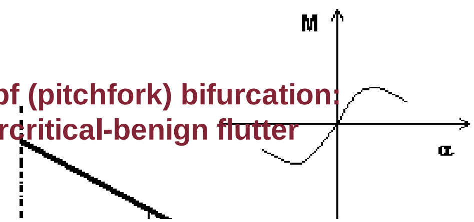


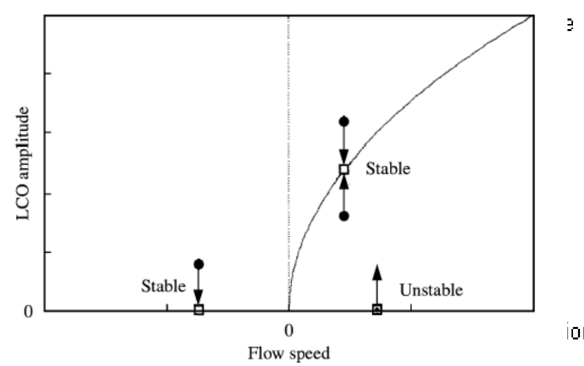

- Nonlinear structural stiffness arising from large displacement gradients
- Freeplay in the control surfaces

- Hopf (pitchfork) bifurcation:

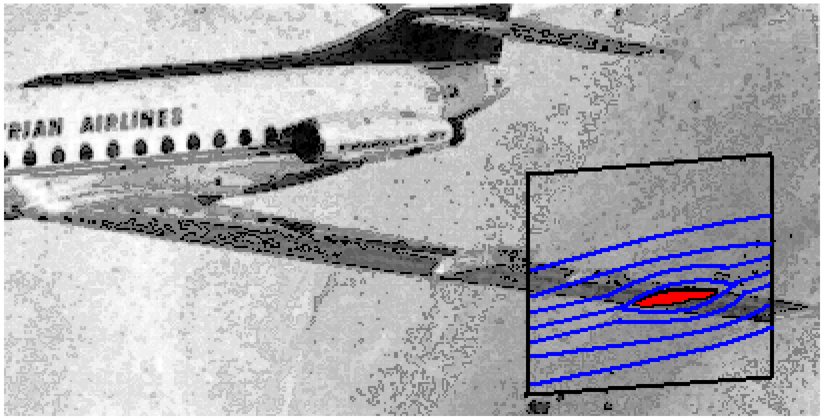


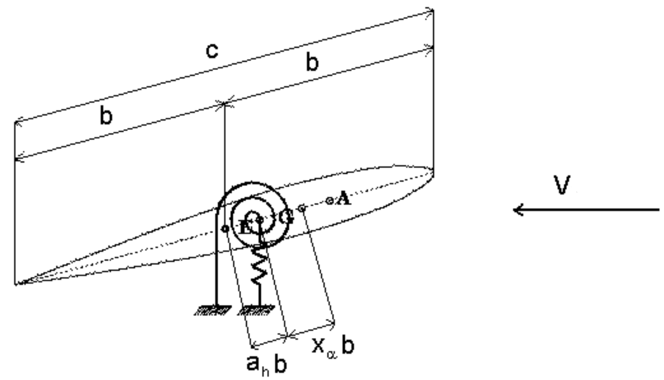

- Hopf (pitchfork) bifurcation: supercritical-benign flutter



# Physical model




**Unsteady potential incompressible unbounded 2D flow**

**hypothesis**

**Nonlinear structure with torsional and bending modes of vibration.**

Uniform and constant flow horizontal velocity
+ vertical deterministic gust

Vibrating structure
with nonlinear (soft) torsional spring

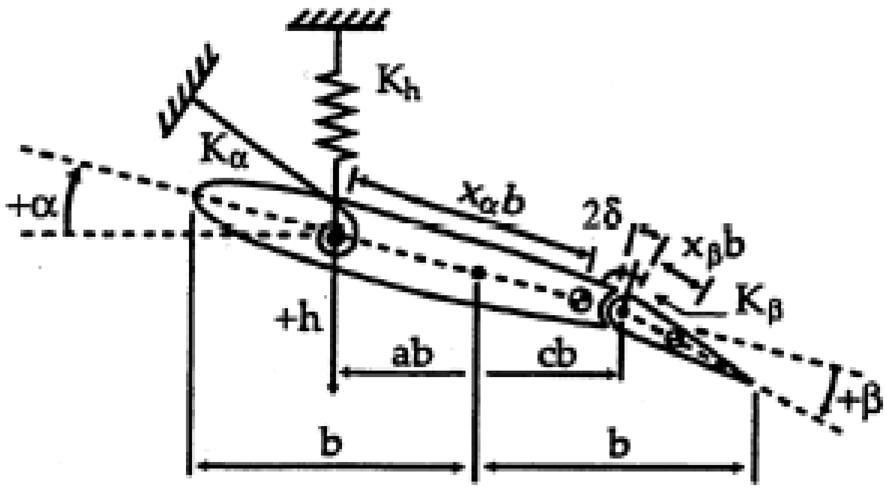
***DOFs***

**Aerodynamic augmented states**

**Heave & Pitch**

# Methematical model:

## A simple (very used ) aeroelastic model:
## Typical Section With Control Surface




- A three degree of freedom aeroelastic typical section with a trailing edge control surface

NONLINEAR TERMS

$$\ddot{\xi} + x_\alpha \ddot{\alpha} + x_\beta \ddot{\beta} + \frac{\Omega_1^2}{U^2}\xi = \mathcal{L}(\xi,\alpha,\beta)$$

$$x_\alpha \ddot{\xi} + r_\alpha^2 \ddot{\alpha} + (r_\beta^2 + (c-a)x_\beta)\ddot{\beta} + \frac{r_\alpha^2}{U^2} M_\alpha(\alpha) = \mathcal{M}_\alpha(\xi,\alpha,\beta)$$

$$x_\beta \ddot{\xi} + r_\beta^2 \ddot{\beta} + (r_\beta^2 + (c-a)x_\beta)\ddot{\alpha} + \frac{r_\beta^2\Omega_2^2}{U^2} M_\beta(\beta) = \mathcal{M}_\beta(\xi,\alpha,\beta)$$

$$\xi = \frac{h}{b},\quad x_\alpha = \frac{S_\alpha}{mb},\quad x_\beta = \frac{S_\beta}{mb},$$

$$\omega_h^2 = \frac{K_h}{m},\quad \omega_\alpha^2 = \frac{K_\alpha}{J_\alpha},\quad \omega_\beta^2 = \frac{K_\beta}{J_\beta},$$

$$r_\alpha^2 = \frac{J_\alpha}{mb^2},\quad r_\beta^2 = \frac{J_\beta}{mb^2},\quad \mu = \frac{m}{\rho\pi b^2}$$

## APPLICATION #1 based on NF theory: control the LCO amplitude and "taming" of explosive flutter

- Nonlinear aeroelastic system with a nonlinear control

$$\frac{d\mathbf{x}}{dt} = \breve{\mathbf{A}}\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{s}(\mathbf{x})$$

$$\mathbf{s} = \kappa \;\; \mathbf{b}_c(\mathbf{c}_c^{\mathsf{T}}\mathbf{x})^3$$ ← Nonlinear feedback

- Then the previous Hopf analysis can be repeated with a new "closed-loop" nonlinear coefficient:

$$\gamma^{(1)} = \gamma_0^{(1)} + \kappa\gamma_1^{(1)}$$

with $\gamma_1^{(1)} := -3 \; c_{c_1}^2 c_{c_1}^* \, b_{c_1}$

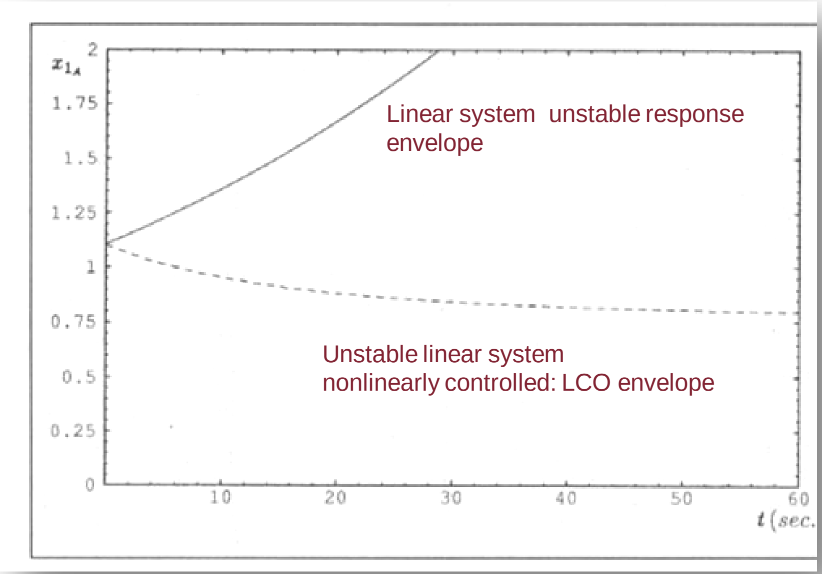
- Condition for **stable limit cycle i**n closed loop conditions →

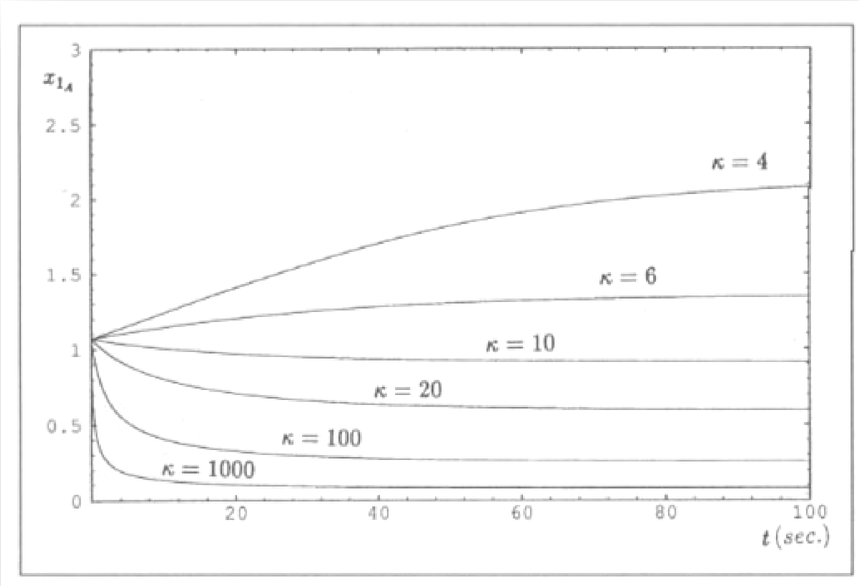
$$\mathcal{R}e(\gamma^{(1)}) = \mathcal{R}e(\gamma_0^{(1)}) + \kappa\,\mathcal{R}e(\gamma_1^{(1)}) \; > \; 0$$

- **NB:** it is always possible to choose κ such as to satisfy previous Eq.

  → it is possibile to control the LCO amplitude by a nonlinear feedback

# APPLICATION #1: LCO nonlinear control




**Unsteable linear system with nonlinear control Different LCO amplitude with different values for non-linear gain κ**

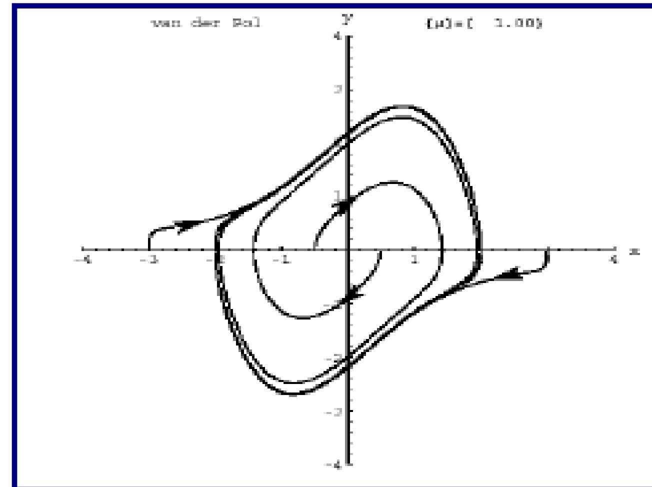
**Different LCO amplitude with different values for non-linear gain κ**

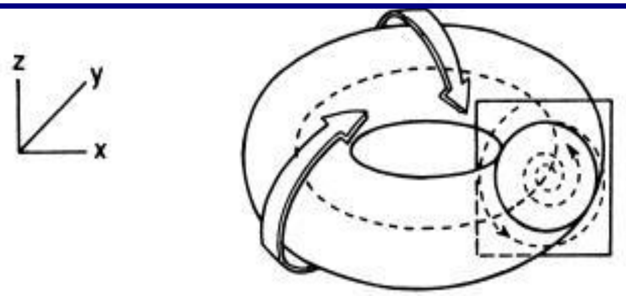
**Ref.: Morino, L., Mastroddi, F., *"Limit-cycle oscillation control with aplication to flutter,"* The Aeronautical Journal, Nov. 1996.**

# APPLICATION #2 based on NF theory:

**periodic, quasi periodic, or chaotic solution by transforming the original problem with different ρ**

- Periodic orbits
  - Harmonic Limit Cycle Oscillations



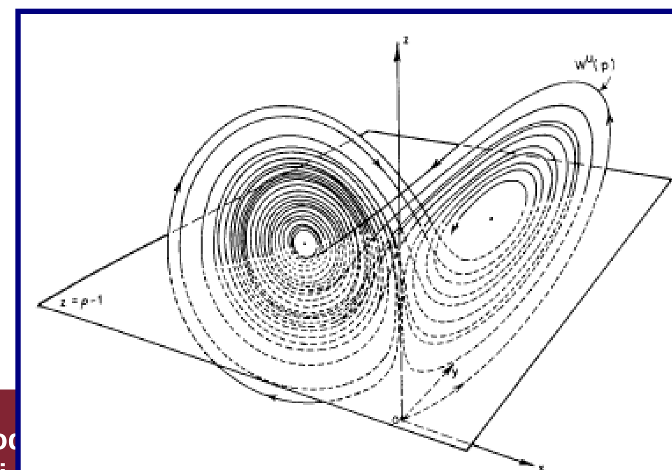




(a)

3-d Flow

- Quasi Periodic orbits
  - Non-periodic Oscillations

- Strange Attractors
  - Chaos

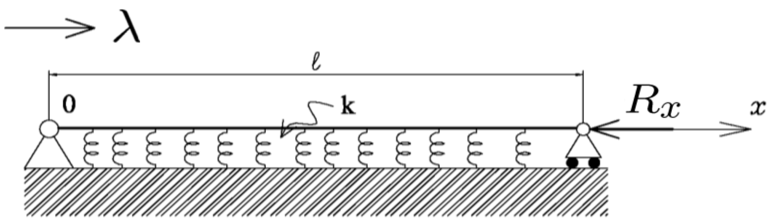


# <u>APPLICATION #2</u> based on NF theory:

## periodic, quasi periodic, or chaotic solution by transforming the original problem with different ρ

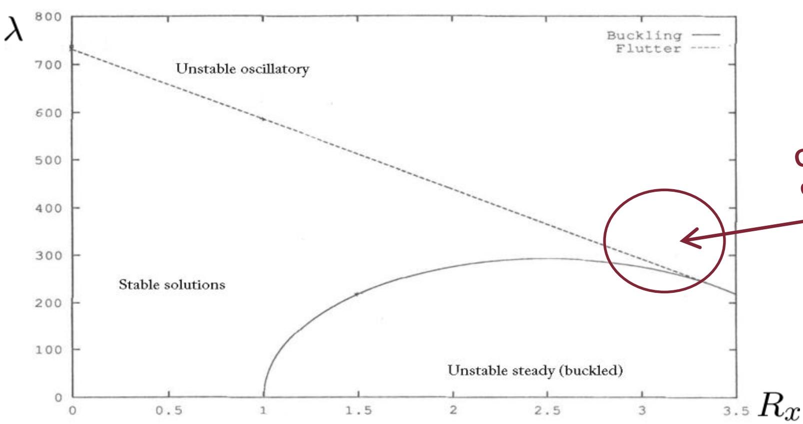
## "Classic" nonlinear Panel flutter →

**Simply supported panel in a supersonic flow with dynamic pressure λ with a buckling load $R_x$ and structural stabilizing nonlinearities**

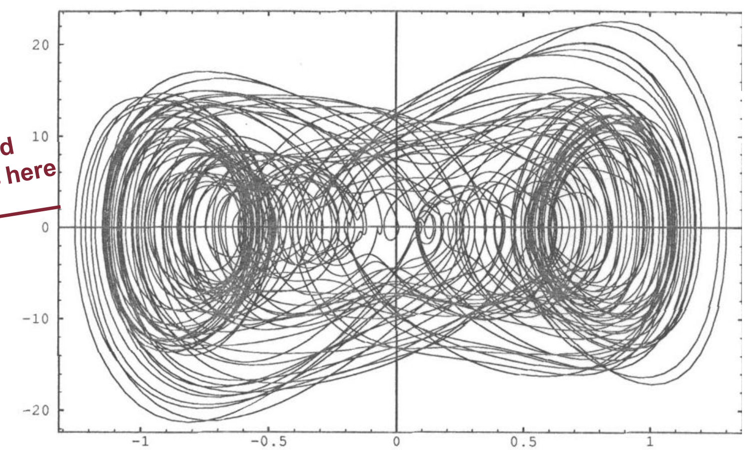



**PDE system solved "a la Galerkin" assuming for the solution** $w(x,t) = \sum_{m=1}^{N_M} w_m(t) \sin\left(\frac{m\pi x}{a}\right)$

$$\ddot{w}_n + g_n \dot{w}_n + [\pi^4(n^4 - n^2 R_x)] w_n + \lambda \sum_{p=1}^{N_M} e_{np} w_p + \sum_{p,q,r=1}^{N_M} c_{npqr} w_p w_q w_r = 0 \qquad n = 1, 2, ..., N_M$$



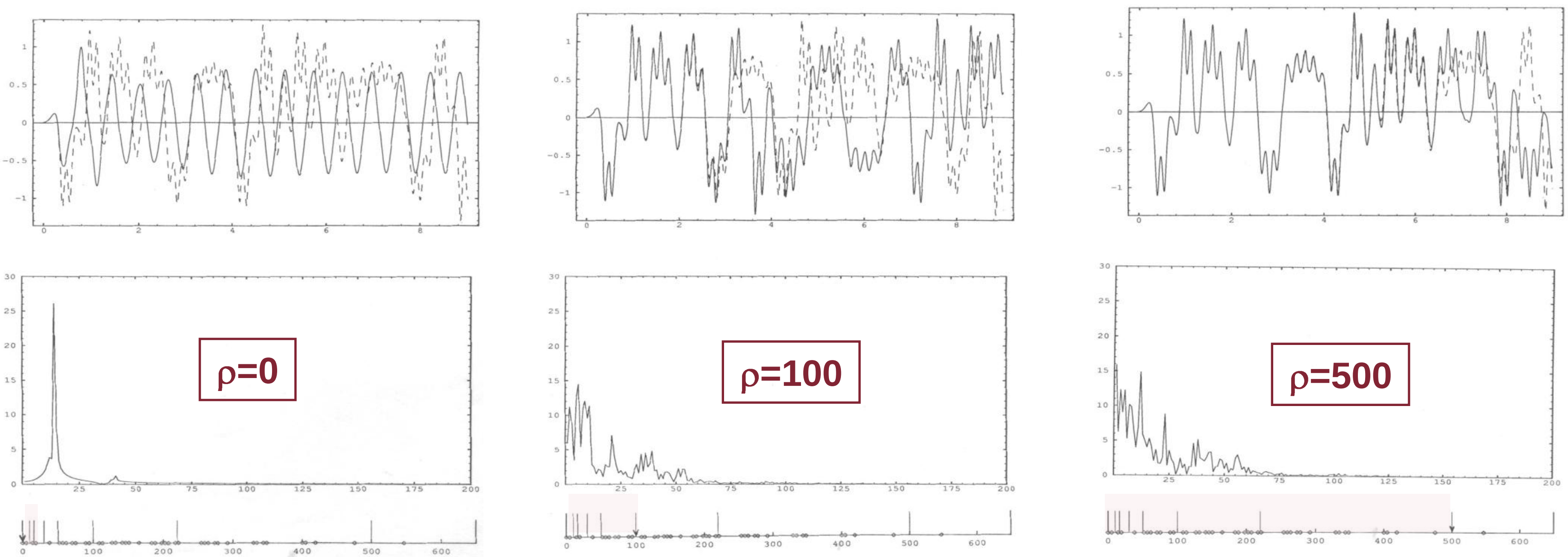

Quasi-periodic and chaotic solutions here



# APPLICATION #2 based on NF theory:

## periodic, quasi periodic, or chaotic solution by transformin the original problem with different ρ

## Nonlinear Panel flutter




- The greater is the assumed radius ρ, the less simply harmonic the solution is
- → the NF is able to identify the NL terms which are responsible of chaotic solution

**Ref.: Morino, M., Mastroddi, F., Cutroni, M.,``Lie Transformation Method for Dynamical System having Chaotic Behavior," *Nonlinear Dinamics,* Vol. 7, No.4, June 1995, pp. 403-428.**

# Some theoretical hightlights on (fifth order) NORMAL FORM (NF) approach

***-A tool for nonlinear analysis of aeroelastic system with polynomial nonlinearities***
***- …. For more complicated bifurcation …. Pre-critical instabilities ….***

**Ali H. Nayfeh, Method of Normal Forms, Wiley Series in Nonlinear Science**

**Ref:** ***Dessi, D., Morino, L., Mastroddi, F., ``A Fifth-Order  Multiple-Scale Solution for Hopf Bifurcations," Computers and Structures,  Vol. 82, 2004, pp. 2723-2731.***

# Fifth-order NF Normal Form (very brief details)

- Considering the fourth order terms the following equations need to be simplified by the NF procedure:

$$\dot{\mathbf{y}} = \mathbf{\Lambda y} + \epsilon^2 \mathbf{g}^{(2)}(\mathbf{y}) + \epsilon^4 \mathbf{h}^{(4)}(\mathbf{y}) + O(\epsilon^6)$$

- Again the NF-method consists of searching for a new state space coordinates through the “near-identity” transformation

$$\mathbf{y} = \mathbf{v} + \epsilon^4 \mathbf{s}^{(4)}(\mathbf{v}) + O(\epsilon^6)$$

- The transformed dynamical system will be in the form

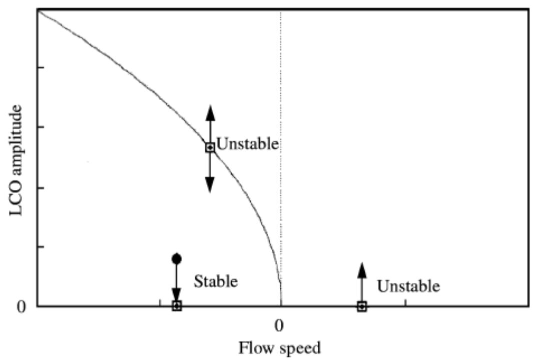
$$\dot{\mathbf{v}} = \mathbf{\Lambda v} + \epsilon^2 \mathbf{g}^{(2)}(\mathbf{v}) + \epsilon^4 \mathbf{g}^{(4)}(\mathbf{v}) + O(\epsilon^6)$$

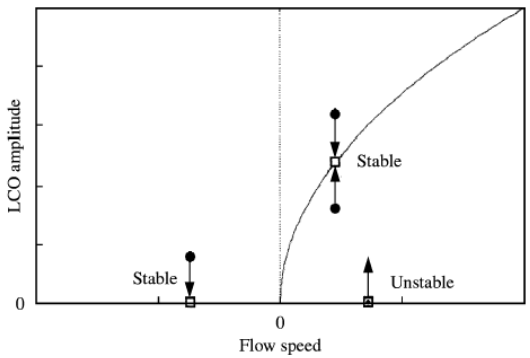
- By using the coordinate transformation $\mathbf{u} = \mathbf{y} = \mathbf{v}$

$$\dot{u}_n = \sum_{p \in \mathbf{I^n_p}} \Psi_{np} u_p + \sum_{pqr \in \mathbf{I^n_{pqr}}} \Sigma_{npqr} u_p u_q u_r + \sum_{pqrst \in \mathbf{I^n_{pqrst}}} E_{npqrst} u_p u_q u_r u_s u_t$$

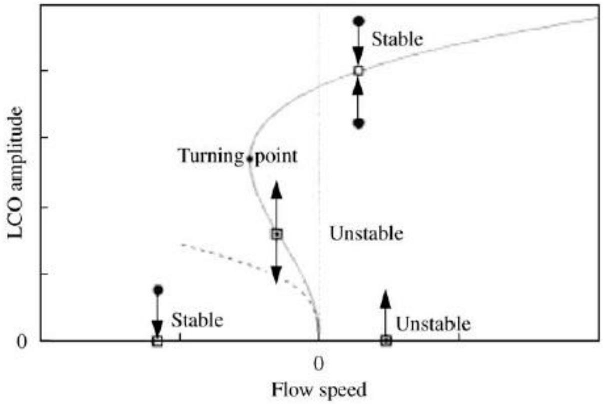
# Loss of Stability for Nonlinear Aeroelastic Systems

- *Eigenvalue crossing of the imaginary axis* (linear analysis)
- *Pitchfork/Hopf bifurcation*: **supercritical (benign flutter)** and **subcritical (explosive flutter)**




- ***Beyond the Hopf bifurcation:***
  ***"knee" bifurcation (precritical LCO's)***



# Aeroelastic interpretation

Turn… d

Flow Speed > Flutter speed

Unsteady Load

**Fluid (linear)**

**Structure (NONlinear)**

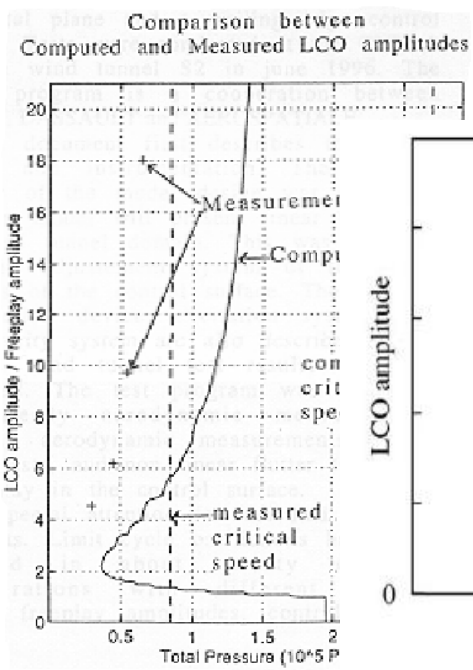
Elastic Displacement

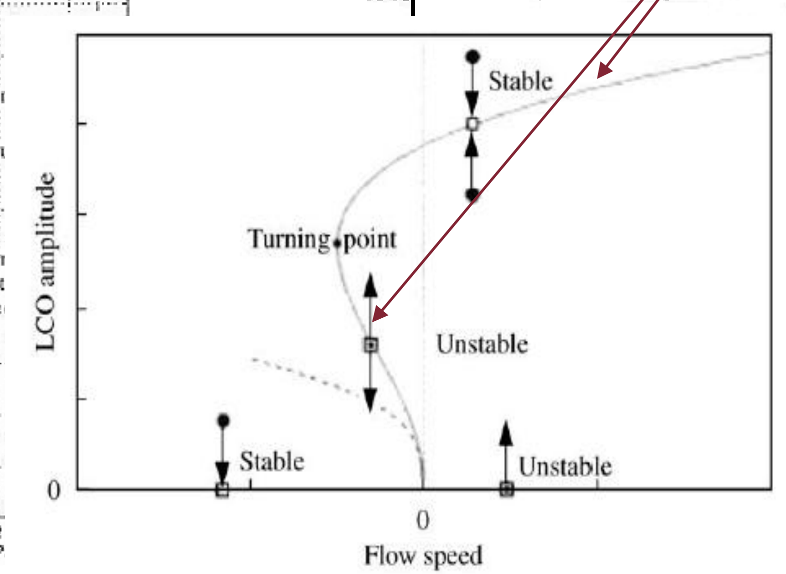
UNStable & Stable …es

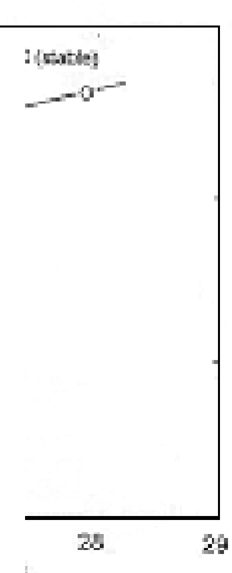
Stable Limit Cycles

- Wind tunnel model with free-play
- Wind tunnel model with transonic effects

- Hopf (knee) bifurcation: “subrcritical to supercritical”

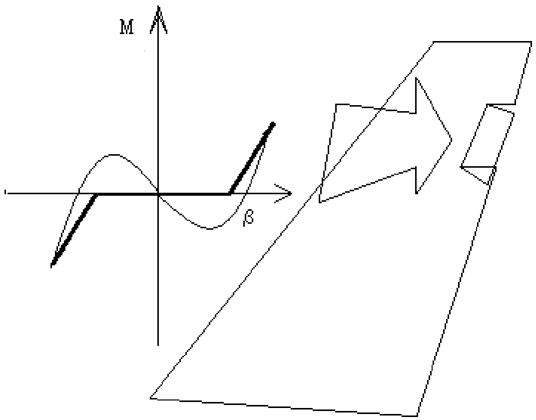


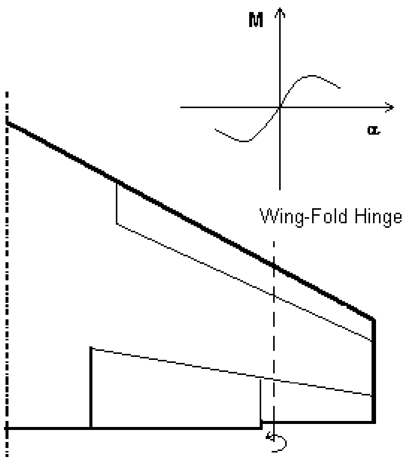



# Physical Sources of Structural Nonlinearities in the Fixed Wing Model

- Freeplay in the control surfaces - bilinear stiffness due to loosely connected structural components (Conner et al., 1997)




- Nonlinear structural stiffness arising from large displacement gradients (Lee, B.H.K., et al., 1989)




- **Approximation** with Polinomial Nonlinearities

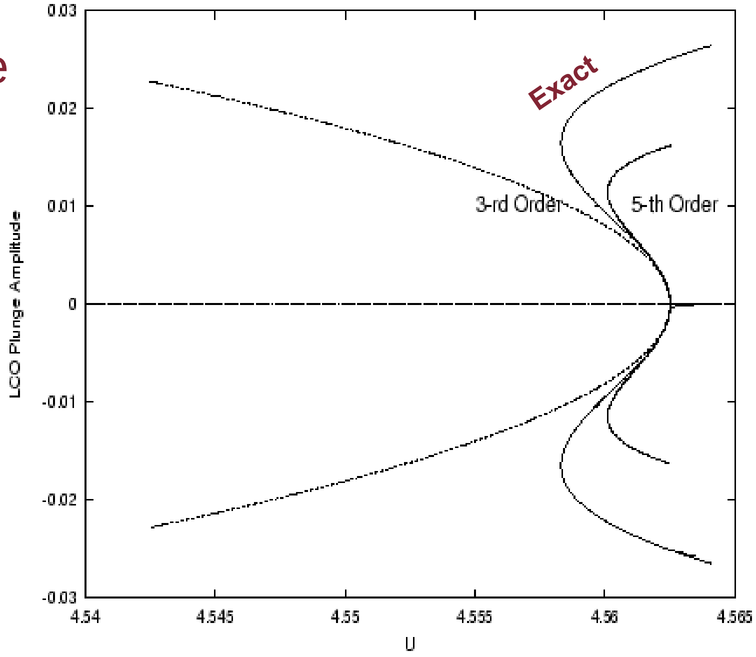
$$M_\alpha = c_{1\alpha}\alpha + c_{3\alpha}\alpha^3 + c_{5\alpha}\alpha^5 \dots$$
$$M_\beta = c_{1\beta}\beta + c_{3\beta}\beta^3 + c_{5\beta}\beta^5 \dots$$

- 8 states variables (aerodynamic ones included) $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$

# Nonlinear Analysis: 3 order v.s. 5 order NF using ρ=0

$$M(\alpha) = \alpha - c_3\alpha^3$$

No **3 order NF analysis** is not able to describe the special kind of bifurcation

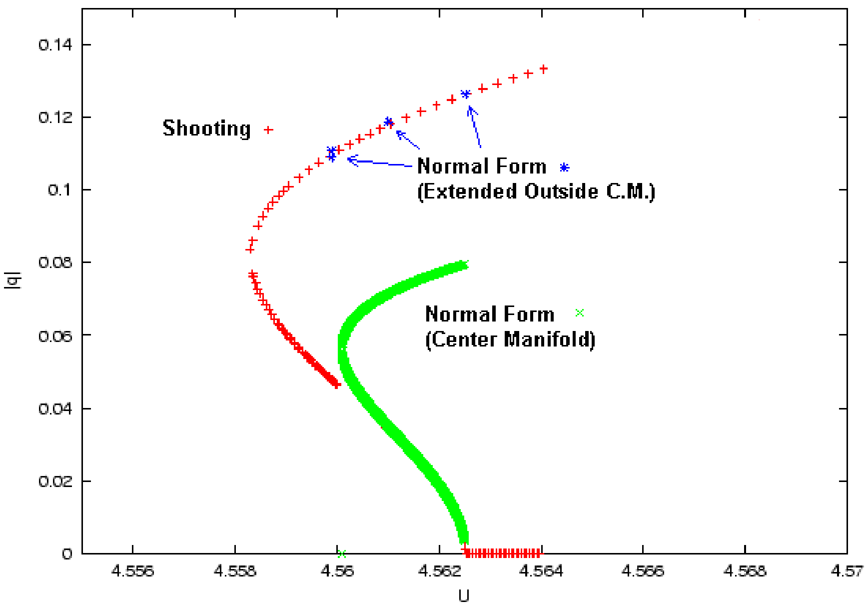


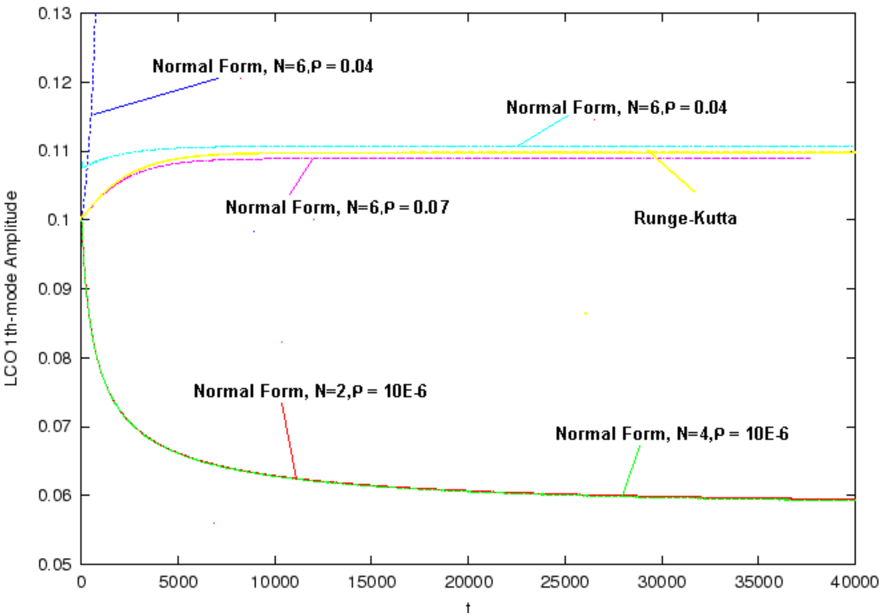

The NF-method has to be extended to **fifth-order** to capture at least the qualitative behavior of the aeroelastic LCO in the case of **‘knee’ bifurcations**.

# Nonlinear Analysis: 5° order NL using ρ>0

$$M(\alpha) = \alpha - c_3\alpha^3$$

• → **To improve the accuracy**, more equations and terms have to be added. Including the near-resonant terms, an extended (in the sense of Center-Manifold) NF-method is implemented

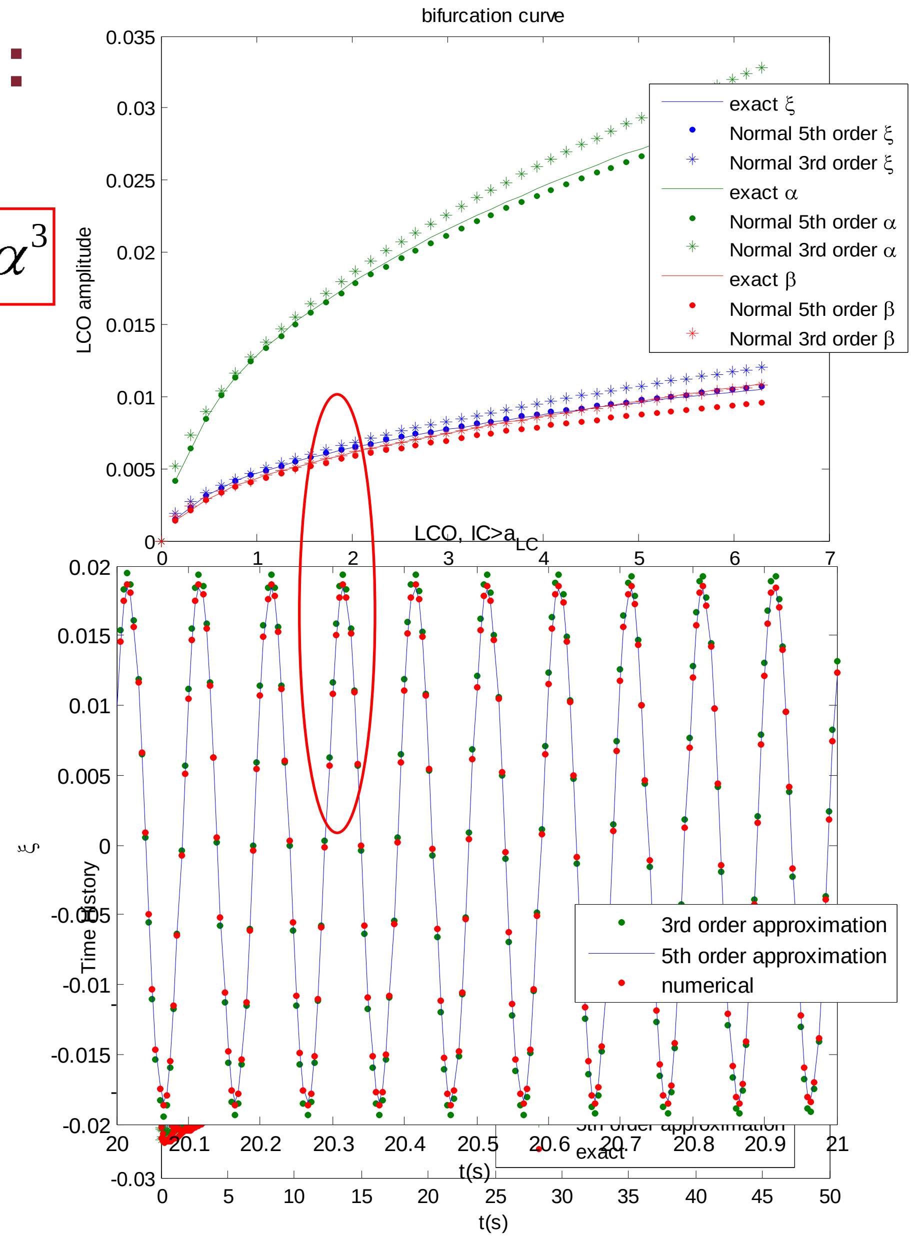


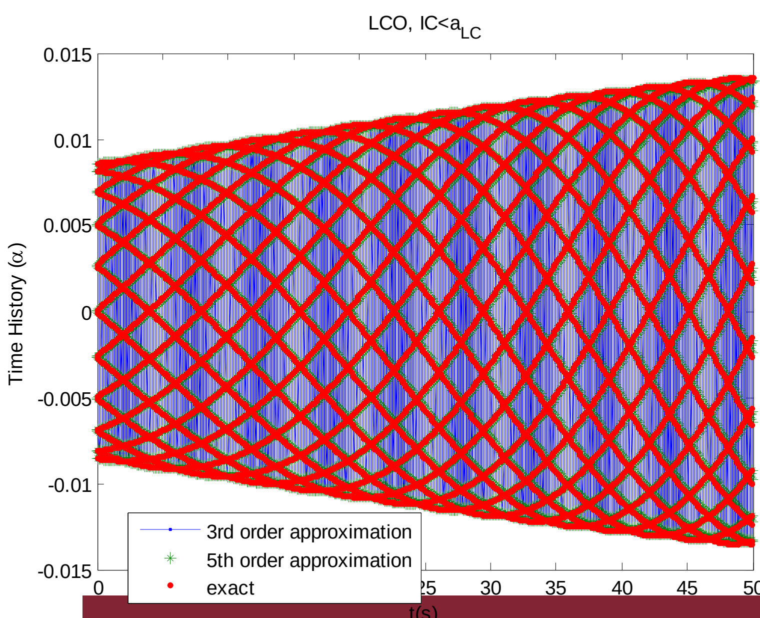



# A detail on this analysis:

$$M(\alpha) = \alpha + c_3\alpha^3$$

- Third order and fifth order normal form (CM) provide a satisfactory approximation

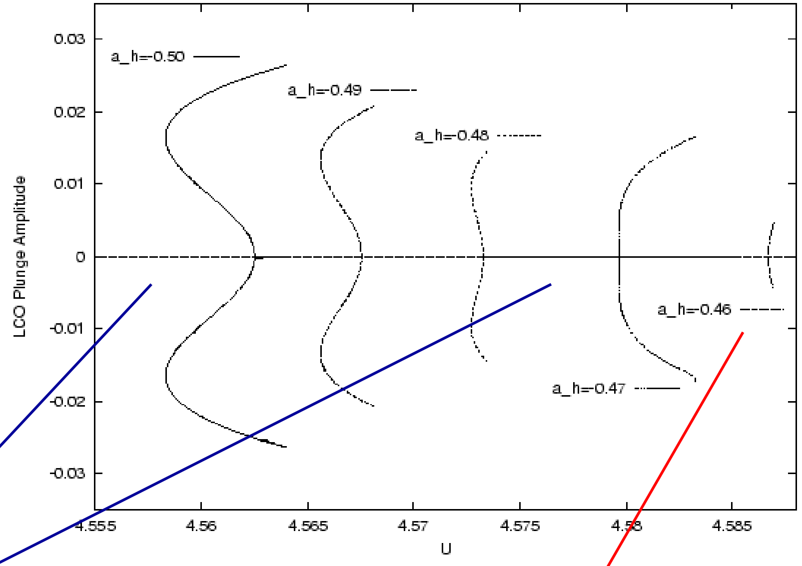


# As sub-product of this analysis:
## identified a parameter moving the system from Hopf to knee-Bifurcation

$$M(\alpha) = \alpha - c_3\alpha^3$$

Plunge Bifurcation Diagrams
for different elastic axis
positions $a_h$




Fisically: the elastic axis position evealed to havea clear influence on the type of bifurcation, *subcritical knee-like* bif. or *supercritical pitchfork* bif.

# NF METHODOLOGY/APPROACH

**-** The nonlinear analysis for **gust-excited** problem

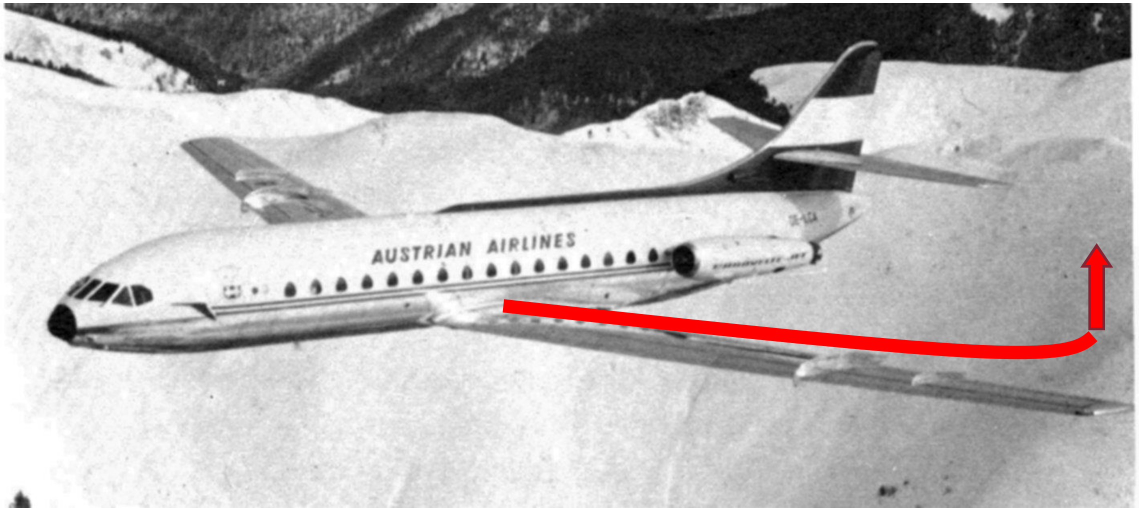
Dessi, D., Mastroddi, F. , "A nonlinear analysis of stability and gust response of aeroelastic systems, *Journal of Fluids and Structures*, 24 (3), p.436-445, Apr 2008

## CASE A:

- Initial conditions
- no gust excitation

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(U)\,\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{g}(\tau;U) \\ \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

## THIS IS THE CASE PRESENTED BEFORE!

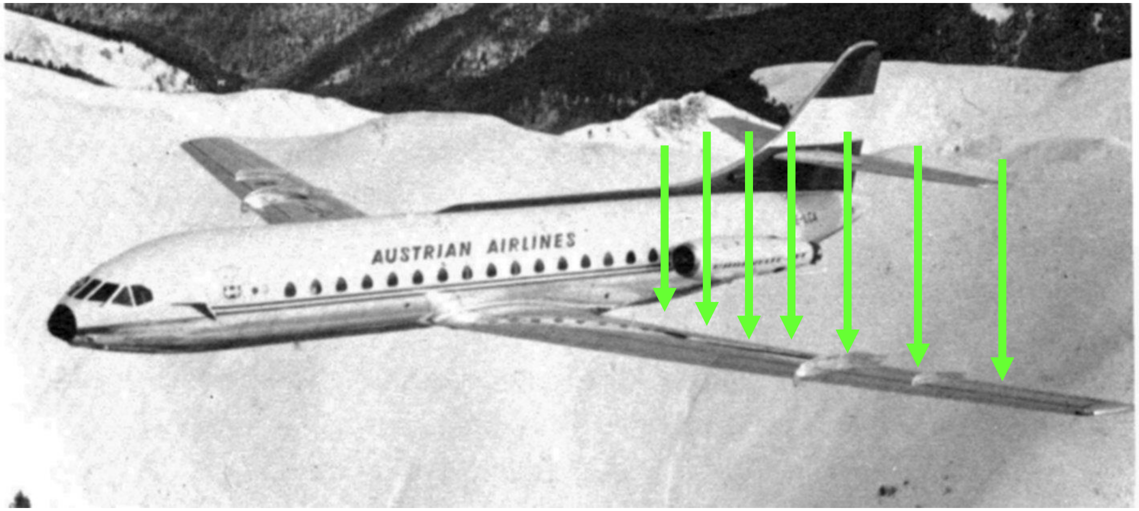


# CASE B: APPLICATION #3

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(U)\,\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{g}(\tau;U) \\ \mathbf{x}(0) = 0 \end{cases}$$

- No initial conditions
- "discrete"

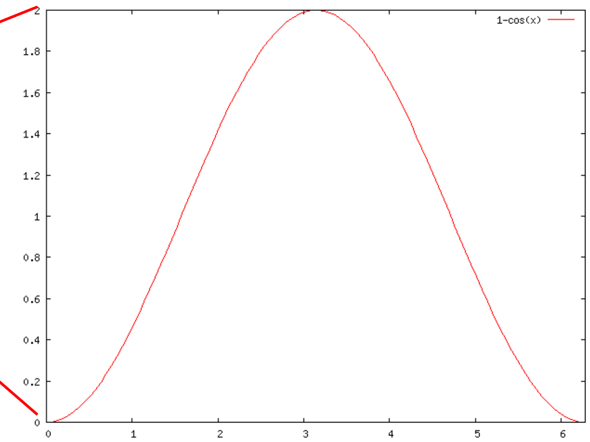
(=time impulsive) gust excitation



# CASE B: Gust excitation with no IC

**In the case of the gust problem, the system dynamics is excited only by the "discrete" gust profile with zero initial conditions**

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}(U)\mathbf{x} + \mathbf{f}(\mathbf{x}) + \mathbf{g}(\tau;U) \\ \mathbf{x}(0) = 0 \end{cases}$$

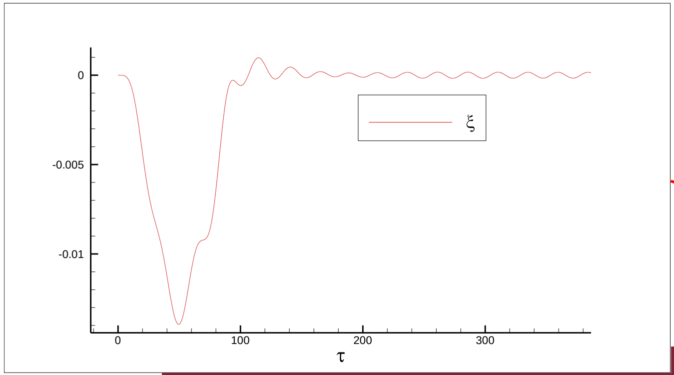


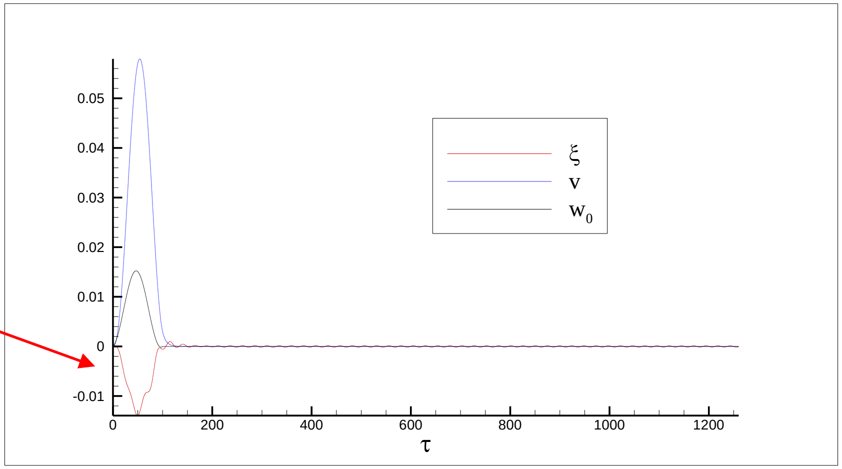

**It is assumed a discrete gust of the wave form**

$$w_G(\tau) = \frac{w_0}{2}\left[1 - Cos\,\frac{\pi\,\tau}{\tau_G}\right]$$

**where:**

- **Gust intensity**
- **Gust gradient**

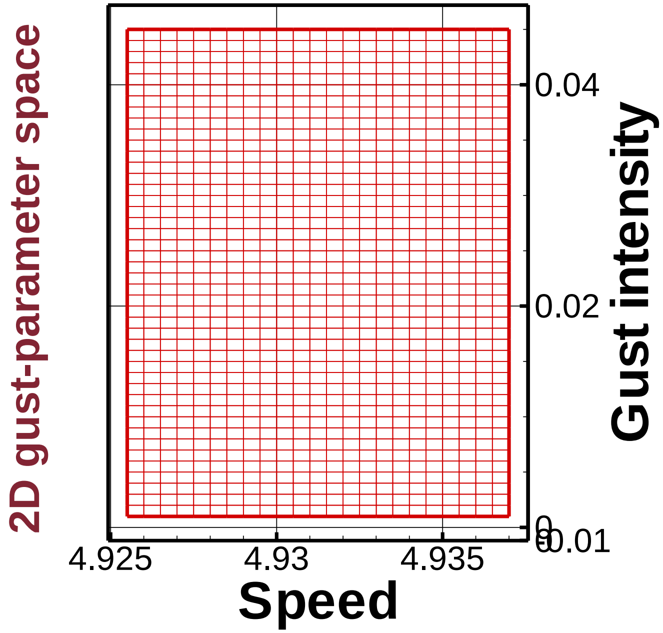


# Gust analysis

• **Fix** the **gust gradient** $\tau_G$ to a value

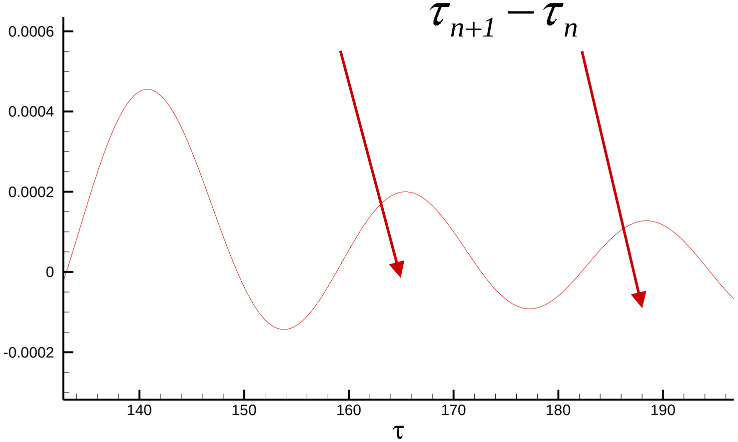
• Define a region in the plane of the parameters and consider **a matrix of numerical simulations** for each value of the grid nodes




• **Introduce the logarthmic damping coefficient** between consecutive peaks in the amplitude modulated periodic solution

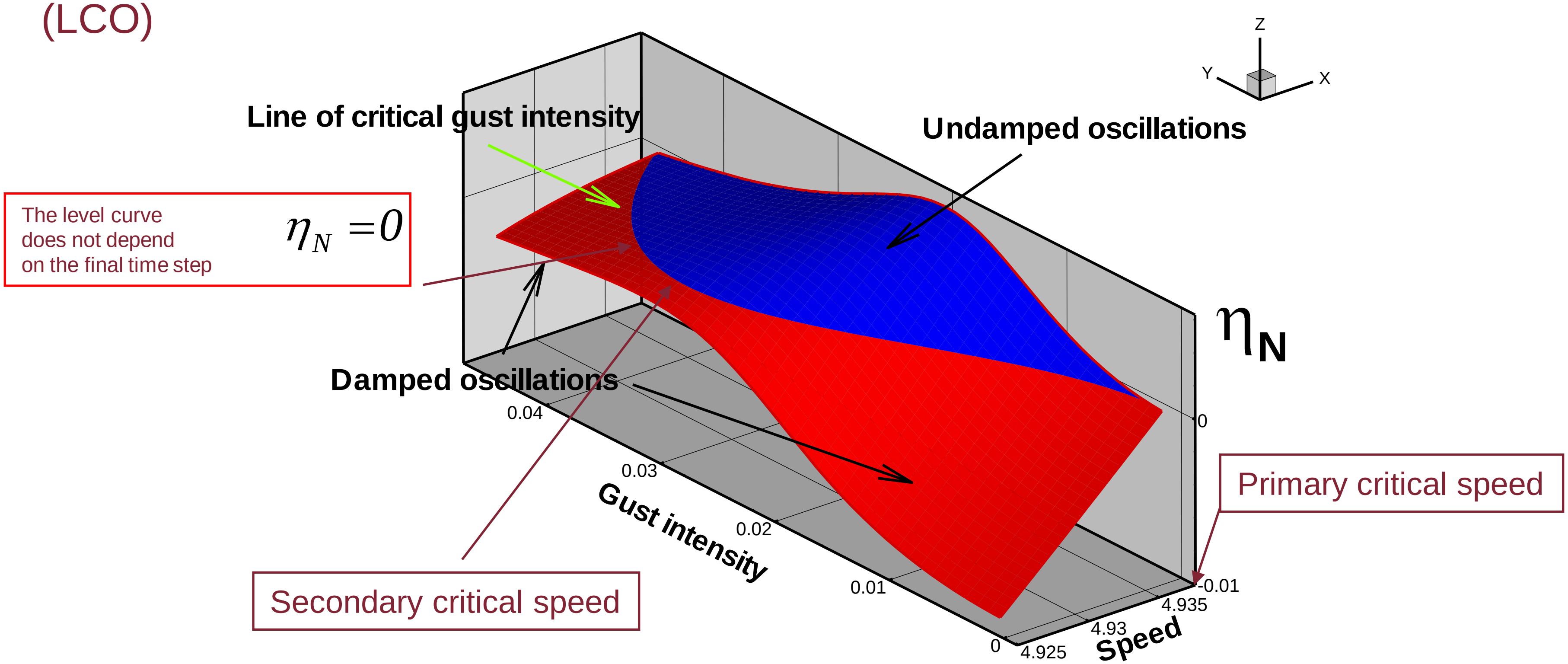
→

$$\eta_{n+1} = \log \frac{x_i^{(max)}(\tau_{n+1}) - x_i^{(max)}(\tau_n)}{\tau_{n+1} - \tau_n}$$



# Gust analysis

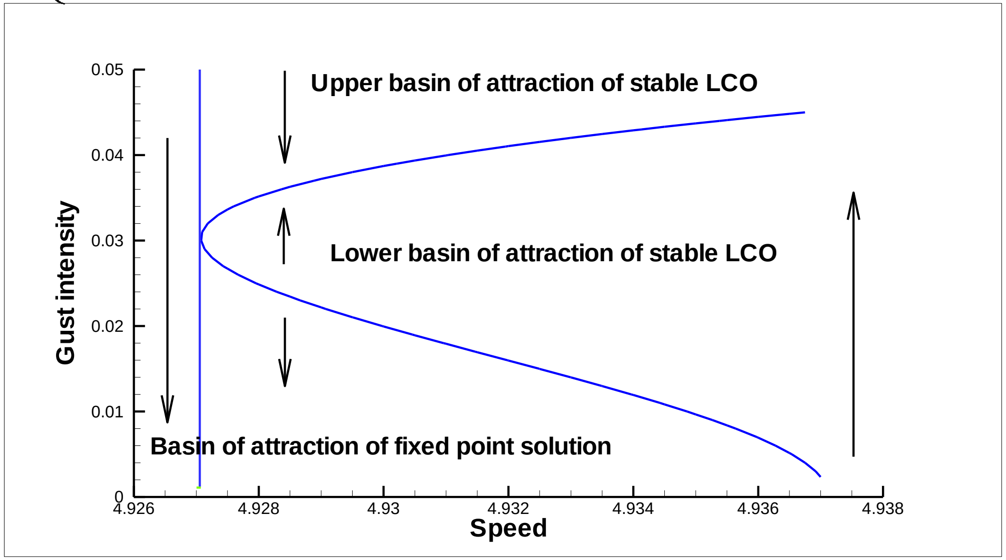
- For each run, consider the logarithmic damping $\eta$ for $n = N$, and obtain the response surface $\eta_N = \eta_N(U, w_0; \tau_G)$
- The level curve $\eta_N = 0$ can be determined by interpolation between the grid nodes
- It represents the **set of parameters (gust intensity for a given flight speed) that leads the state-vector to be very close to a stable or unstable periodic solution** (LCO)



# Basin of attraction: critical gust intensity

• The branch of the curve $\eta_N = \eta_N(U, w_0; \tau_G)$ for which $\partial\eta_N / \partial w_0 > 0$ across it, represents the critical gust intensity $w_0^{(c)} = w_0^{(c)}(U)$ characterized by the property that

$$\begin{cases} w_0 > w_0^{(c)} \Rightarrow \text{the solution is undamped} \\ w_0 < w_0^{(c)} \Rightarrow \text{the solution is damped} \end{cases}$$




• Thus, a **basin of attraction** for the solution has been identified in the space of physical parameters.

•This analysis has to be **repeated for several gust gradients** $\tau_G$

## … and what happens if the nonlinearities are not polynomials?....

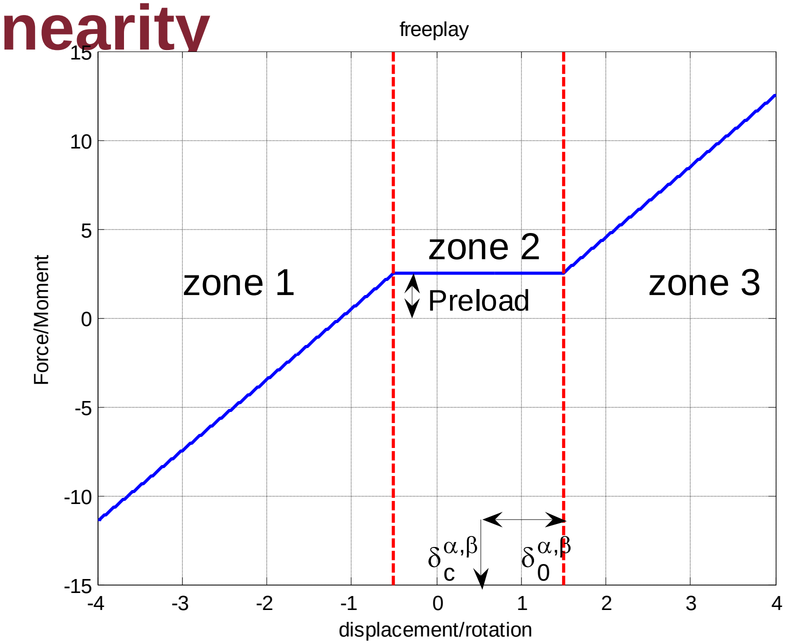
***For example*: How is important to correctly model the discontinuity of a freeplay? (i.e., avoiding polynomials)**

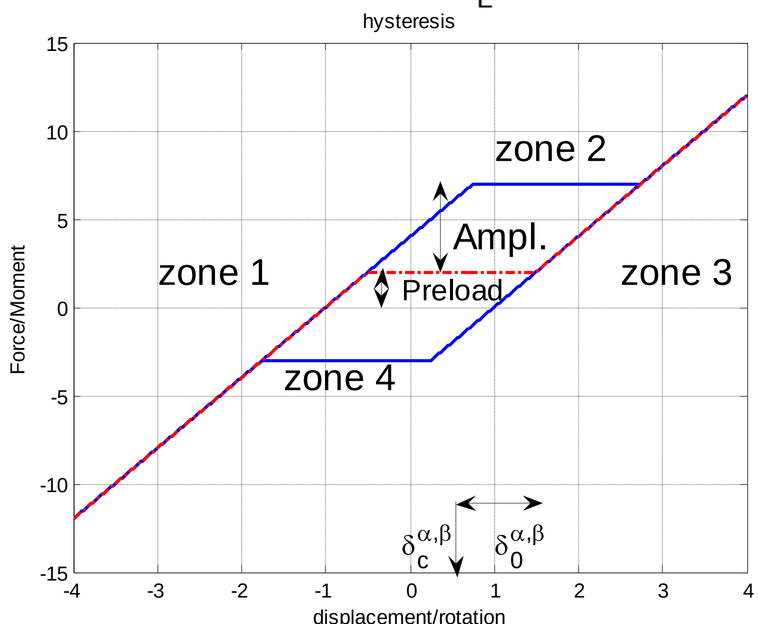
# APPLICATION #5: physically more realistic modeling for Freeplay/Hysteresis Nonlinearity

- DOMAIN SUB-DIVISION: Zone 1 & Zone 3

$$\mathbf{M}_{gg}\ddot{\mathbf{q}} + \mathbf{K}_{gg}^{OUT}\mathbf{q} = \mathbf{f}_{left/right}$$

$$\mathbf{f}_{left/right} = \mp[\mathbf{0}\ \mathbf{M}^{-1}\ \mathbf{0}]\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{r_\alpha^2}{U^2}c_{1\alpha}(\delta_0^\alpha - \delta_c^\alpha) \\ \frac{r_\beta^2\Omega_2^2}{U^2}c_{1\beta}(\delta_0^\beta - \delta_c^\beta) \\ 0 \\ 0 \end{bmatrix}$$



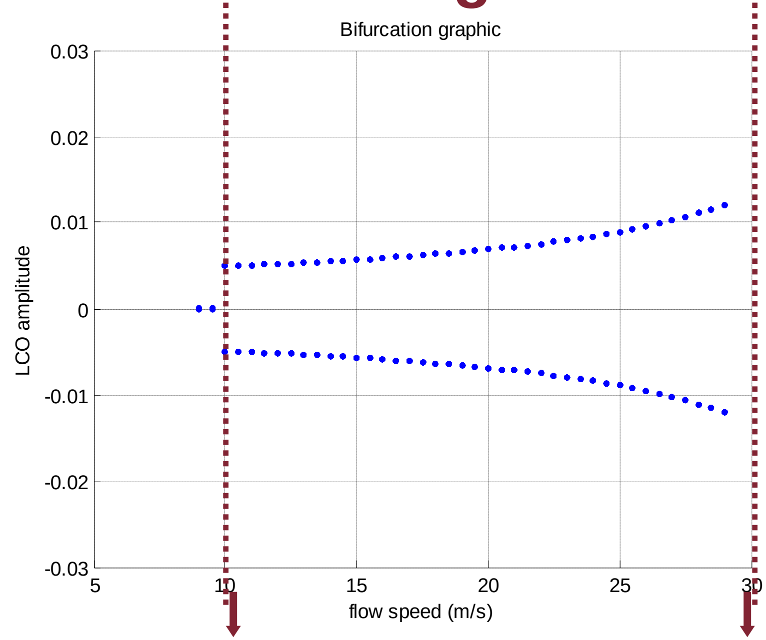

- Zone 2 & Zone 4

$$\mathbf{M}_{gg}\ddot{\mathbf{q}} + \mathbf{K}_{gg}^{IN}\mathbf{q} = \mathbf{f}_{up/down}$$

$$\mathbf{f}_{up/down} = \mp[\mathbf{0}\ \mathbf{M}^{-1}\ \mathbf{0}]\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \frac{r_\alpha^2}{U^2}(preload \pm amplitude) \\ \frac{r_\beta^2\Omega_2^2}{U^2}(preload \pm amplitude) \\ 0 \\ 0 \end{bmatrix}$$

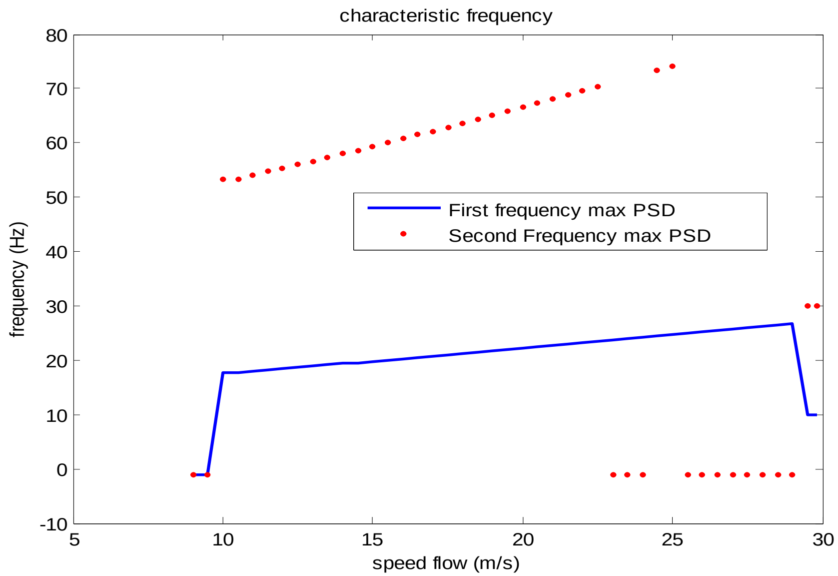
# Freeplay Results: direct numerical integration

- Amplitude of LCOs increases with flow speed and activated between the two linear stability limit



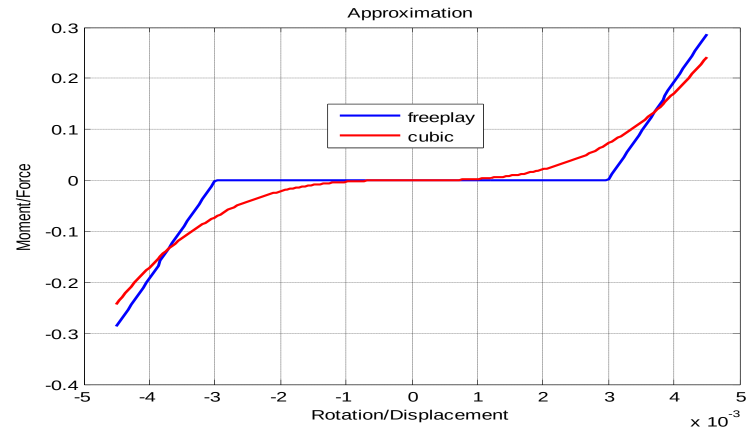

(linear) Flutter speed flap disconnected

(linear) Flutter speed flap connected

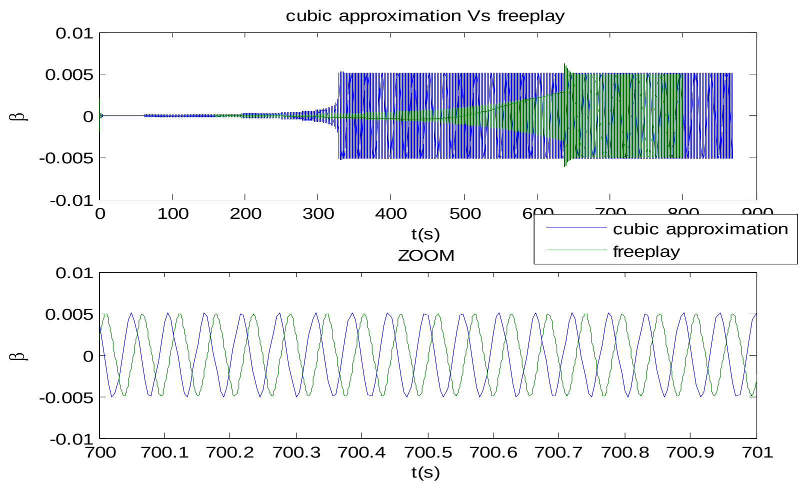



- Second Frequency characteristic 3 times the first

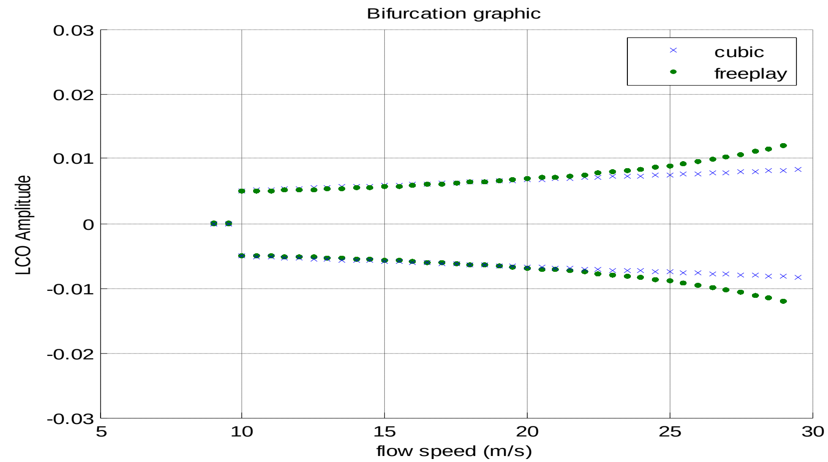
# How to apply Normal Form in this case? (1)




- Freeplay **must be approximated** by a polinomial nonlinearity in order to perform the NF analysis→



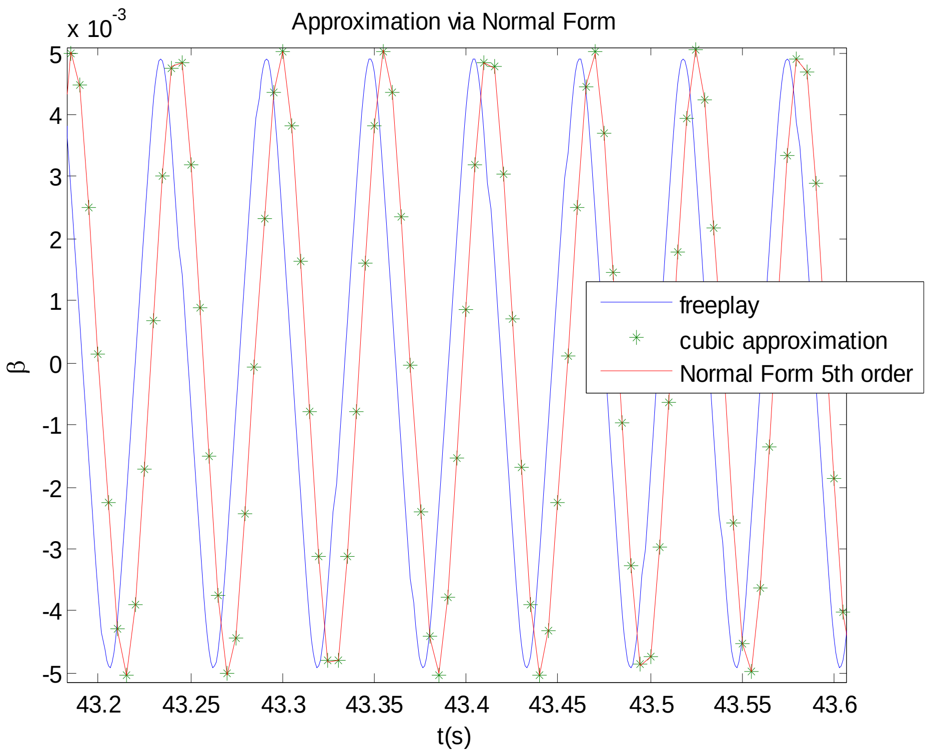

- The **cubic approximations** give satisfactory results for the amplitude and the frequency of LCOs. However the transitory is quite different

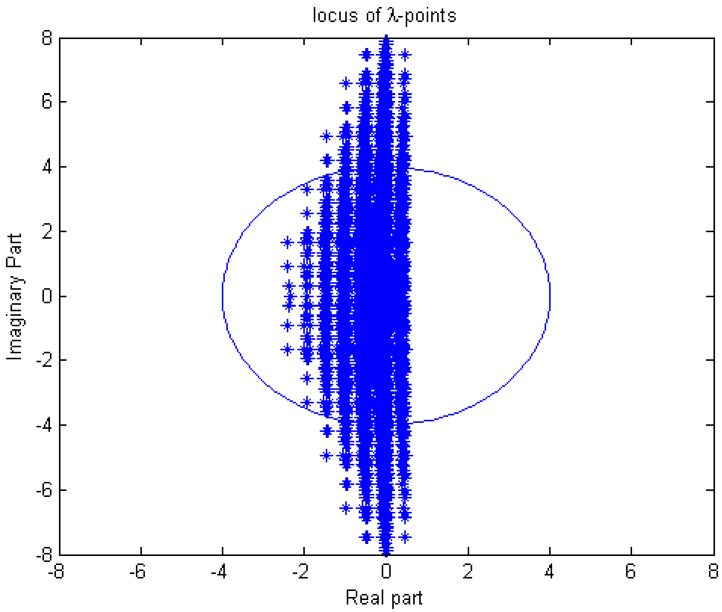



- All the polynomial modeling for the freeplay discontinuity works locally well for LCO (around bifurcation point)
- …but …

# How to apply Normal Form in this case? (2)

- → all the equations must be used (ρ<4) to obtain a correct approximation for LCO



# CONCLUDING REMARKS 1/2 (from ourown experience in nonlinear aeroelasticity)

- Once a system can be mathematically modeled by **nonlinear first order differential equation with polynmial nonlinearities** (many aeroelastic systems can be modelled so), NF approach may represent a local powerfull tool
  - To obtain the **analytic solution around a Hopf bif.**
  - To **reduce** the system size to a restricted set equations (Center Manifold theorem)
  - To study the LCO stability in **precritical** condition (with a higher order analysis (basin of attraction defined with I.C. or with suitable input)
  - To find a **nonlinear feedback** to “tame” linear and nonlinear oscillations
  - To identify the **nonlinear contributions** responsible of chaotic behavior

## CONCLUDING REMARKS 2/2

- If system nonlinearities **are not in a polynomial form** (it is the case of the freeplay modeling)
  - A **polynomial approximation of the nonlinearity** could be used and the NF approach can efficiently capture the nonlinear behavior of the system if near-resonace terms are included
  - An **extension of the NF theory** should be developed (something is existing like Lie Transformation) in this case

*<u>Comment</u>*

The freeplays nonlinearities can be **trivially** identified but **not so easily analyzable by** NF approach

Polynomial nonlinearity are (**not so-trivially**) analyzable by NF approach but are **not identifieable** by actual measurements at all